Joint work with Chen:

- X smooth proj curve $/ k$.
- Higgs bundle of rank n over X is (E, θ)

w/ E: V.B. of rank n / X.

$\theta : E \longrightarrow E \otimes \Omega^1_X$, $\mathcal{O}_X$-linear map.

$\mathcal{M} = \{ (E, \theta) \}$: Hitchin mod stack

(Here it's more elementary. If we use "space" instead of "stack" then we get a mod. sp. corresp. to the sp. of reps of $\pi_1 X$

Hitchin map: $h : \mathcal{M} \longrightarrow \mathcal{A} := \bigoplus_{i=0}^{n} H^0(X, S^i \Omega_X)$

$(E, \theta) \mapsto (a_1, \dots, a_n)$

where $a_i = \mathrm{tr}(\Lambda^i \theta : \Lambda^i E \to \Lambda^i E \otimes S^i \Omega_X)$

on the trace formula side, $\mathcal{M}$ is a kind of home trace formula, and the Hitchin fibres is the space of conjugacy classes on space of conjug. over integrals So h is the geometrisation of the trace formula.

Main features of h: • Abelian fibration

$a \in \mathcal{A}$ generic, then $h^{-1}(a)$ is an abelian var up to connected component and inertia. (stacky business). we want to understand this abelian fibration phenomena in higher dim, not going to any singular fibers.

The way Hitchin understood it is very elementary but elegant via:

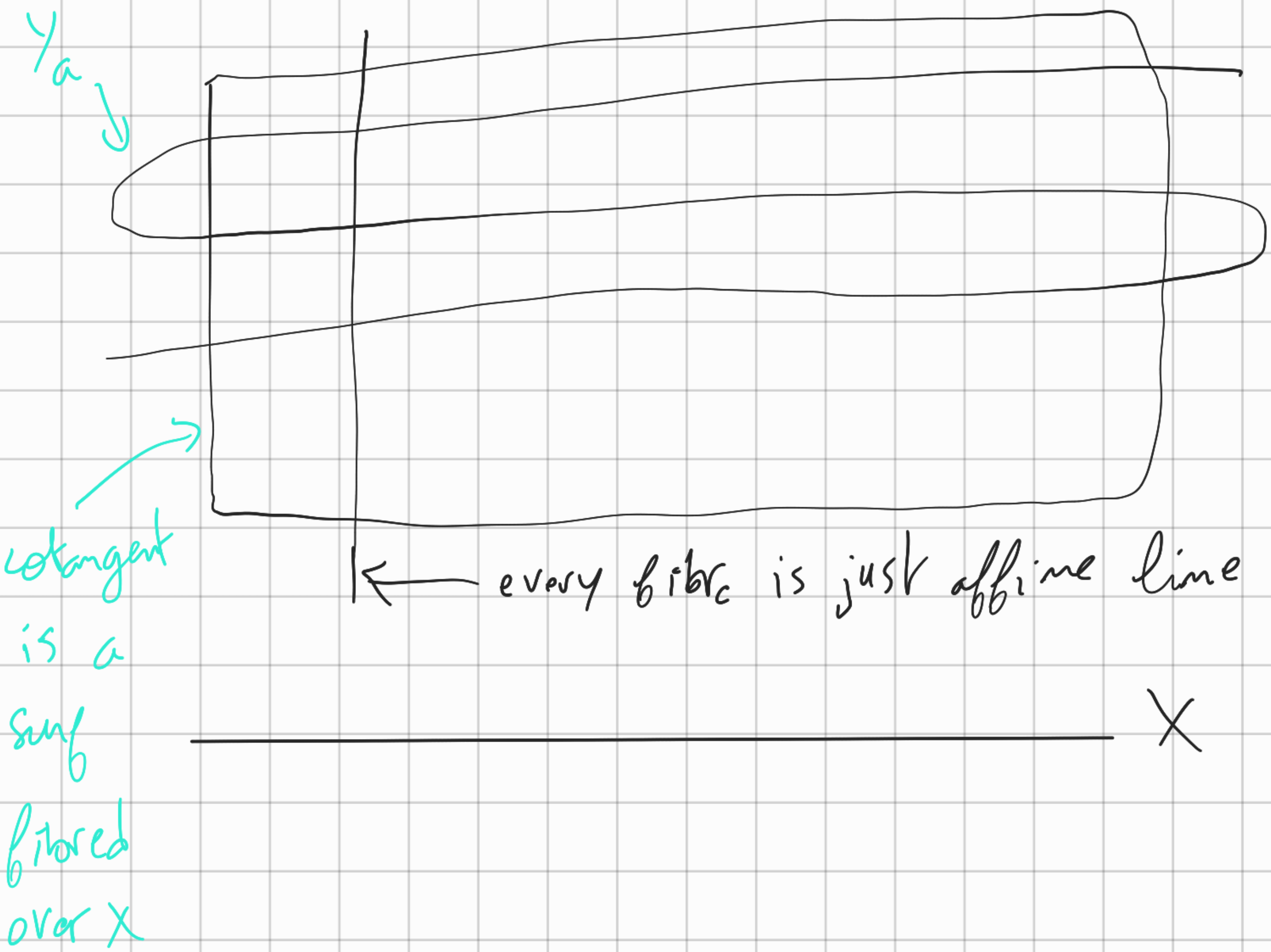
Spectral curves:

So how can we prove the main feature in the case of curves?

$\forall\, a \in \mathcal{A} \rightsquigarrow Y_a$ curve $C\ |\Omega_X|$

$$\begin{array}{c} Y_a \\ \downarrow \quad n\text{-fold covering} \\ X \end{array}$$

defined by $t^n - a_1 t^{n-1} + \dots + (-1)^n a_n$



Y_a is a 3-fold covering over X and

t is the vertical coordinate

Cayley-Hamilton Theorem (CH)

$$\mathcal{M}_a \xleftrightarrow{\text{identified}} \left\{ \begin{array}{l} F : \mathcal{O}_{Y_a}\text{-module s.t.} \\ E = p_{a*} F \text{ is v.B. of rank } n \end{array} \right\}$$

use θ to define E as a module over the cotangent, then CH theorem says that this module is supported on the spectral curve

if Y_a is smooth then:

$$\mathcal{M}_a = \{\text{invertible } \mathcal{O}_{Y_a}\text{-module}\} \quad \text{for } \dim X = 1$$

$$\big((\mathrm{Jac}(Y_a) \times \mathbb{Z}\big) / \mathbb{G}_m$$

$\forall\, a \in \mathcal{A}$, in general:

(✱) $P_a = \mathrm{Pic}(Y_a) \curvearrowright \mathcal{M}_a$ for any spectral curve

so the fibres are not abelian varieties but they are acted on by some abelian variety, so this picture should be generalized for every group.

$\nexists$ spectral cover for a general group. For classical groups, we see spectral curves as involutions [13:55] but this theory is just case by case. so in general,- by general there are Donagi's cameral covers.

<u>Cameral covers</u>: G a reductive $/ \mathbb{C}$.
char$(k) \nmid$ the order of the weyl group.

$$\mathfrak{g} := \mathrm{Lie}(G) \circlearrowleft G$$

chevalley restriction th. $\longrightarrow$ $\underbrace{k[\mathfrak{g}]^G}_{\substack{\text{ring of } G\text{-inv} \\ \text{funs on Lie} \\ \text{algebras}}} \simeq \underbrace{k[\mathfrak{t}]^W}_{\substack{\text{ring of } W\text{-inv} \\ \text{funs on Cartan} \\ \text{algebras}}} = k[a_1, \dots, a_m]$

$a_1, \dots, a_m$ are not canonical and are homogeneous G-inv polys of degree $d_1, \dots, d_m$. (up to ± 1 they could be canonical)

Let's translate this into stacky language which allow us to put all this construction into a very transparent footing.

$$
\begin{tikzcd}
\mathfrak{g}/G \arrow[r] & \mathfrak{g}/\!/G = \operatorname{Spec}(k[\mathfrak{g}]^G) =: ⬠ \arrow[d, equal, "\text{Chevalley's th.}"'] \\
 & \mathfrak{t}/W
\end{tikzcd}
$$

alg. stack quotient

$\circlearrowright$ by homothety $\mathbb{G}_m$

$\underbrace{\mathfrak{g}/\!/G}$ affine space

(we add the space to support the stack somehow)

$\circlearrowright$ $\mathbb{G}_m$

$$t\cdot(a_1,\dots,a_m) = (t^{d_1}\cdot a_1, \dots, t^{d_m}\cdot a_m)$$

So here you have transparent construct. of h.

So

$$
\begin{tikzcd}
X \arrow[r, "(E,\theta)"'] \arrow[rr, bend left] \arrow[dr, "{E,\,\Omega_X}"'] & \mathfrak{g}/G \times \mathbb{G}_m \arrow[r, "m"'] \arrow[d] & ⬠/\mathbb{G}_m \arrow[d] \\
 & BG \times \mathbb{G}_m \arrow[r] & B\mathbb{G}_m
\end{tikzcd}
$$

cameral cover

generalises spectral curves in a nice way

$$\begin{tikzcd} \tilde{X}_a \arrow[r] \arrow[d] & \mathfrak{t}/\mathbb{G}_m \arrow[d] \\ X \arrow[r, "a"'] & \mathfrak{c}/\mathbb{G}_m \end{tikzcd}$$

Last word in the case of $\dim_X = 1$:

<u>Regular centraliser</u>:

$$\begin{tikzcd} I^{reg} \arrow[d] & & I \arrow[d] \\ \mathfrak{g}^{reg} \arrow[r, hook] \arrow[dr, "\chi^{reg}"'] & \mathfrak{g} \arrow[d] & J \arrow[dl, dash] \\ & \mathfrak{c} & \end{tikzcd}$$

$x \in \mathfrak{g}$, $I_x = \{ g \in G \mid \mathrm{ad}(g)x = x \}$

I^{reg} is a smooth abelian grp scheme.

<u>Lemma</u>: $\exists$ smooth abelian grp scheme $J \longrightarrow \mathfrak{c}$

with G-equiv. $(\chi^{reg *})J \simeq I^{reg}$,

Moreover, this extends to a $!$ homo: $\chi^* J \longrightarrow I$. $(\star_1)$ [25:00]

$a : X \longrightarrow \mathfrak{C}$, $J_a := a^* J$

$P_a := \{ J_a\text{-torsors over } X \}$

So this P_a plays the role of the P_a stack in $(\star)$

(when you don't have a spectral curve, you have to go through this grp-scheme construction, you have the P_a in $(\star)$ and the hom in $(\star_1)$ gives the action $P_a \circlearrowright \mathcal{M}_a$.

when the cameral curve $\tilde{X}_a$ is smooth, then this action is simply transitive, and P_a is an abelian variety up to π_0 and inertia)

studied via geometric endoscopy theory

duality

(when G becomes G^* then π_0 become inertia and vice versa)

Now we look at $\dim_X > 1$

Now X is some smooth proper alg. var.

If you reduce to $\dim_X = 2$, we don't loose much since the π_X of a higher dim. var. can be reduced to a $\pi_1 X$ for $\dim_X = 2$ (cutting by hyperplane doesn't change the π_0). Same thing for Higgs bdl, so case of surface is the crucial case)

(E, θ) Higgs bdle., E : rk m V.B. $/X$

$\theta : E \longrightarrow E \otimes \Omega^1_X$, $\underbrace{\theta \wedge \theta = 0}$

integ. condition which makes the study difficult.

$$\theta : E \otimes \underbrace{T_X}_{(\Omega^1_X)^*} \longrightarrow E \qquad \mathcal{O}_X\text{- linear}$$

Locally, $E = \mathcal{O}_X^m$, $T_X = \bigoplus_{i=1}^{d} \mathcal{O}_X \partial_{x_i}$

θ is just the map of how ∂_{x_i} act on E.

$$\theta(\partial_{x_i}) = \theta_i \in \mathrm{End}(\mathcal{O}_X^m)$$

$$[\theta_i, \theta_j] = 0 \qquad (\text{from } \theta \wedge \theta = 0)$$

So basically θ is the commuting twisted endos.

So in higher dim case:

<u>target stack</u>: Let G be reductive grp of Lie$(G) =: \mathfrak{g}$

$$\underbrace{\mathcal{C}}_{\text{stack}} = \Big\{ \theta : \underset{\substack{\text{v.s. of} \\ \text{dim } d \text{ (kind of tangent space)}}}{V} \longrightarrow \mathfrak{g} \;\Big|\; [\theta(v), \theta(v')] = 0 \Big\} \Big/ \underset{\text{adjoint action}}{G}$$

e.g.: $G = GL_n$, $d = 2$, then if you rigidify V to be $\mathbb{A}^2$ then: $\mathcal{C} = \{\theta_1, \theta_2 \in \mathfrak{gl}_n \mid [\theta_1, \theta_2] = 0\} / (GL_2 \times G)$

So

Higgs field:

$$\begin{tikzcd} X \arrow[r, "{(E,\theta)}"] \arrow[dr, "{(\star_3)\ (\Omega_X, t)}"'] & \mathcal{C} \arrow[d] \\ & BGL_d \times BG \end{tikzcd}$$

So giving a Higgs bdl is described by the space of maps from the surface X to some kind of target stack $\mathcal{C}$ (some space of commuting matrices, and divide by whatever natural action they have, adj. action on $\mathfrak{g}$ and base change action on the vector space $GL_d \times G$

After twisting $(*_3)$, the G-action gives rise to the G-bdl we start with (E), and the GL_d-action ——— cotangent bdle

Now Simpson studies the Hitchin map in case $G = GL_m$ only.

$$h_X : \mathcal{M}_X \longrightarrow \mathcal{A}_X = \bigoplus_{i=1}^{m} H^0(X, S^i\Omega_X)$$

$$(E,\theta) \longmapsto (a_1, \dots, a_m), \quad a_i = tr(\theta^i : E \to E \otimes S^i\Omega_X)$$

but now it's much more complicated because in $\dim_X = 1$, Ω_X is a line bdle so the S^i are also line bdles (it's just tensors).

but now in higher dim, Ω_X is a rank d V.B., so the $\mathrm{Sym}^i \Omega_X$ are complicated.

<u>Problem now:</u> describe h_X, the Simpson-Hitchin map. Is it an abelian fib.?

<u>general answer</u>: No

<u>partial Answer</u>: $\mathcal{M}_X \xrightarrow[\text{abelian fibration}]{} \mathcal{A}_X \xrightarrow[\text{closed embedding}]{} \mathcal{B}_X$

geom. of $\mathcal{A}_X$ seems to be connected to classification of surfaces.

Spectral data: for $G = GL_n$.

let E : n-dim. V.S.

$[\theta_i, \theta_j] = 0$

$E = k[x_1, \dots, x_m]$ - mod. of fin. length n

x_i acts as θ_i

[42:00]

$$E = \bigoplus_{\alpha \in k^m} E_\alpha$$ (E_α supported by the point $\alpha \in k^m$, $k = \bar{k}$)

$\downarrow$

a cycles $\sum_\alpha \lg(E_\alpha)\,\alpha$. $\in (\mathbb{A}^d)^{(n)}$

length

$(\mathbb{A}^d)^n / S_d$

this construction works / any ring R.

let $C = \{\theta_1, \dots, \theta_d \in \mathfrak{g} \mid [\theta_i, \theta_j] = 0\}$

$$\big(\underbrace{C /\!/ G}_{\text{not known to be reduced.}}\big)^{red} = t^d /\!/ W \longrightarrow C / G$$

$\exists$ nat. action $t^d /\!/ W$

the purple map is enough for our geom. construction to work, although it's not as strong as reducedness of $\mathbb{C}/\!/G$.

So $h_X : \mathcal{M}_X \longrightarrow \mathcal{A}_X = Ch_m(T^*X/X)$

it mean that on every fiber you pick out some 0-dim. cycles and you want them to vary algebraically on X and this construction, every time you have a Higgs bdle, then you have a relative Chow class. then:

abelian fib. like
the 1-dim case

$$h_X : \mathcal{M}_X \xrightarrow{\uparrow} \mathcal{A}_X = Ch_m(T^*X/X)$$

$$\mathcal{A}_X \hookrightarrow \mathcal{B}_X$$

closed immersion but we don't know what dimension $\mathcal{A}_X$ is, this is a diff. question.

So why it' a closed embedding? (from invariant theory, follows from Weyl's lemma 1 below)

<u>Lemma 1</u>: (Weyl book)

V, a vector sp. , then

$$\mathrm{Chow}_m(V) \hookrightarrow V \times S^2 V \times \dots \times S^m V$$

$$[v_1, \dots, v_m] \longmapsto (v_1 + \dots + v_m,\ v_1 v_2 + \dots,\ \dots)$$

is a closed embedding (in dim 1 it's an iso)

(check proof: interesting)

<u>lemma 2</u> $\chi : V \times \mathrm{Chow}_m(V) \longrightarrow S^m V$

$$(v, [v_1, \dots, v_m]) \longmapsto (v - v_1) \cdots (v - v_m)$$

$\mathrm{Cayley}_m(v) = \chi^{-1}(0)$ (the vanishing locus of the Cayley-Hamilton polynomials)

then $\mathrm{Cayley}(v) \longrightarrow \mathrm{Chow}_m(v)$ finite but not flat map

$a \in \mathcal{A}_X$

$$\begin{tikzcd}
X_0^{CM} \arrow[dr] \arrow[ddr, bend right, "\text{finite flat}"'] & & \\
& X_a \arrow[r] \arrow[d, "\text{finite, not flat}"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Cayley}_m(T^*X/X) \arrow[d, "\text{finite}"] \\
& X \arrow[r, "a"'] & \mathrm{Chow}_m(T^*X/X)
\end{tikzcd}$$

when X is a surface, $\exists$ nice resolution called Cohen-Macaulay spectral surface X^{CM} which makes the map in orange finite flat.

can $\rightsquigarrow$ prove $P_a = \mathrm{Pic}(X_a^{CM})$, $P_a \curvearrowright \mathcal{M}_{b_a}$ simply transitive

, P_a is an abelian var up to π_0 and inertia.

$\rightsquigarrow$ good theory of spectral surfaces in case of surfaces, we went a step further, not taking a spectral cover, but a slight modification of it using the Cohen-Macaulayfication of the surface and then the Picard var of that stuff is going to describe the Hitchin fibers, it works very well in the abstract, but given any surface you can do the calculations but it's absolutely horrendous, you can do it in some simple surface like abelian surface or fiber surface, but in case of some general surface, I don't know how to do it. But it gives another perspective on the study of alg surf. by studying the geom. of Higgs bdls.

This construction only works for $G = GL_n$, I don't know how to construct spectral data map for groups other than GLm.