The motivating example of Num (X).

main reference: Equivalence classes of polarizations and moduli spaces of sheaves
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recall: From last talk that a Bridgeland stability condition on $D^b(X)$ is a heart $\mathcal{A} \subset D^b(X)$, i.e. a full abelian subcategory with similar properties as $Coh(X)$, together with an additive homomorphism $Z: K_0(X) \longrightarrow \mathbb{C}$. The aim of the seminar is to show that the collection of all such stability conditions, denoted by $Stab(X)$, has a complex manifold structure and moreover there is a collection of walls $\{W_i\}$ on $Stab(X)$ (closed submanifolds with boundary of codimension 1) such that $Stab(X) - \bigcup \{W_i\}$ is a finite collection of chambers.

The aim of this talk is to give a concrete example. More specifically we will treat the case X a smooth algebraic variety over $\mathbb{C}$ of dimension n larger than one. To make the construction more tangible we will only speak of stability conditions coming from polarizations on X. This will allow us to describe $Stab(X)$ in terms of a finitely generated free abelian group and the walls in terms of explicit equations.

I) Stability under different polarizations

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Recall that the exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ defines a map $Pic(X) \to H^2(X, \mathbb{Z})$. If $H^2(X,\mathbb{Z})$ is freely generated then we define $Num(X)$ as $Pic / \ker(Pic(X) \to H^2(X,\mathbb{Z}))$. Clearly, $Num(X)$ is a finitely generated free abelian group. The images of ample invertible sheaves in $Num(X)$ are called polarizations.

def: For a polarization L and a torsion free coherent sheaf E, let

$$\mu_L(E) = c_1(E).L^{n-1} / rk(E).$$

As always we say that E is (semi)-stable w.r.t. L if for all coherent subsheaves $F \subset E$ st $0 < rk(F) < rk(E)$ we have $\mu_L(F) (\leq) \mu_L(E)$.

remark: If $n=2$ or $rk(E)=2$ then it is sufficient to check ss on locally free subsheaves st the corresponding quotients are torsion free.

By fixing the numerical invariants $c_1 \in Pic(X)$ and $c_2 \in A^2_{num}(X)$ the Chow group of cycles of codimension 2 modulo numerical equivalence, let us define the following moduli space:

Let $M_{\mu_L}(c_1, c_2)$ parametrize locally free, (rank two) stable sheaves with fixed c_1 and c_2.

(a quick boundedness argument allows the quot/GIT approach which produces a quasiprojective moduli space.)

proposition (example): Let E be a locally free rank 2 sheaf on a smooth algebraic surface X. Let L_1 and L_2 be two polarizations on X. Suppose that E is L_1-stable and L_2-unstable.

Then there exists an invertible sheaf $\mathcal{O}_X(D)$ on X with $L_1 \cdot D < 0 < L_2 \cdot D$, $\mathcal{O}_X(D + c_1(E))$ divisible by 2 in $Pic(X)$ and $c_1(E)^2 - 4c_2(E) \leq D^2 < 0$.

corollary: Let E be a locally free rank 2 sheaf on a smooth surface X. If E is stable w.r.t. one polarization and unstable w.r.t. another one then $c_1(E)^2 - 4c_1(E) < 0$.

remark: This is a special case of Bogomolov's instability theorem: if $c_1^2 > 4c_2$, then $M_L(c_1, c_2) = \emptyset$ for any polarization on a surface X. Donaldson and Kobayashi showed that if $c_1(E) = c_2(E) = 0$, then E is stable w.r.t some polarization L on X iff E comes from an irreducible unitary representation of the fundamental group $\pi_1(X)$. (Narasimhan-Seshadri generalization).

proposition (example) 2: Let E be a locally free rank 2 sheaf on a smooth n-dimensional variety X. Suppose that E is L_1-stable and L_2-unstable. Then there exists an invertible sheaf $\mathcal{O}_X(D)$ on X, and integers i, j satisfying $0 \leq i < j \leq n-1$ such that,

a surface

$S := L_1^{n-1-j} \cdot L_2^{i} \cdot (L_1 + L_2)^{j-i-1}$ and we have:

D is a divisor i.e a linear combination of things of cod 1

(i) $\mathcal{O}_X(D + c_1(E))$ is divisible by 2 in $\mathrm{Pic}(X)$;

(ii) $[c_1(E)^2 - 4c_2(E)] \cdot S \leq D^2 \cdot S < 0$; $\rightarrow$ reduction to the surface case

(iii) $(D \cdot L_1) \cdot S < 0 < (D \cdot L_2) \cdot S$

II) Equivalence classes and chambers

def: Let L_1, L_2 be two polarizations on X. Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in A^2_{num}(X)$. We define $L_1 \overset{s}{\geq} L_2$ if for every locally free rank two sheaf E with Chern classes c_1, c_2 then E is L_2 stable implies E is L_1 stable. We define $L_1 \overset{s}{=} L_2$ iff $L_1 \overset{s}{\geq} L_2$ and $L_1 \overset{s}{\leq} L_2$

Proposition: Let $\Delta_L = \{ L' \text{ such that } L' \text{ is a polarization and } L' \geq L \}$ (polarizations that at least agree with L)

$\mathcal{E}_L = \{ L' \text{ such that } L' \text{ is a polarization and } L' \overset{s}{=} L \}$ (polarizations that agree with L)

then we have that: (i) $\Delta_{L'} \subset \Delta_L$ iff $L' \overset{s}{\leq} L$

(ii) $\Delta_{L'} = \Delta_L$ iff $L' \overset{s}{=} L$

We already know that $\mathrm{Num}(X)$ is a finitely generated free group. There is an open cone (called Kahler cone) C_X, in $\mathrm{Num}(X) \otimes \mathbb{R}$ which is spanned by polarizations. Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in A^2_{num}(X)$.

Def: (i) Let $S \in A^{n-2}_{num}(X)$, and $\gamma \in \mathrm{Num}(X) \otimes \mathbb{R}$, we define

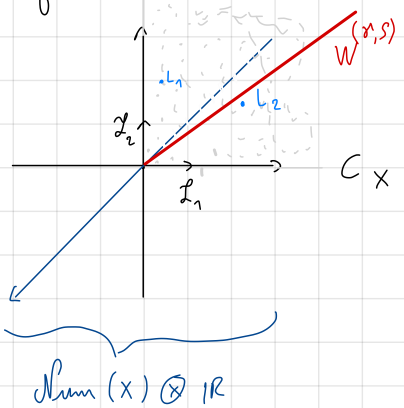
surface

$$W^{(\gamma, S)} = C_X \cap \{ x \in \mathrm{Num}(X) \otimes \mathbb{R} \text{ such that } x \cdot \gamma \cdot S = 0 \}$$

(ii) We define the set of walls $W^{(c_1, c_2)}$ as the collection of elements of the form $W^{(\gamma, S)}$ with S a complete intersection surface in X and γ is the numerical equivalence class of a divisor D on X such that $\mathcal{O}_X(D + c_1)$ is divisible by 2 in $\mathrm{Pic}(X)$ and that

$D^2 \cdot S < 0$, $c_2 + \frac{D^2 - c_1^2}{4} = [Z]$ for some locally complete intersection Z of codim 2

(iii) A wall of type (c_1, c_2) is an element in $W(c_1, c_2)$. A chamber of type (c_1, c_2) is a connected component of $C_X - W(c_1, c_2)$. A $\mathbb{Z}$-chamber of type (c_1, c_2) is the intersection of $Num(X)$ with some chamber of type (c_1, c_2).



$L_2 \cdot \gamma \cdot \varsigma > 0$ and $L_1 \cdot \gamma \cdot \varsigma < 0$ ✓ condition (iii)

$D^2 \cdot \varsigma < 0$ ✓ condition (ii) just have to find an E with fixed Chern classes c_1 and c_2 which

hypothesis ✓ condition (i)

exists because $c_2 + \frac{D^2 - c_1^2}{4} = [Z] \to \left(c_2 + \frac{D^2 - c_1^2}{4}\right) \cdot \varsigma = Z \cdot \varsigma$

$\to \frac{(4c_2 - c_1^2)\varsigma}{4} + \frac{D^2 \cdot \varsigma}{4} = Z \cdot \varsigma$

$\to (4c_2 - c_1^2)\varsigma > 4Z \cdot \varsigma$

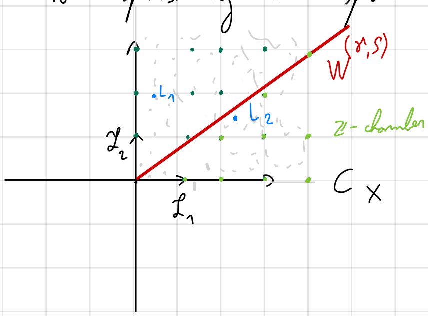
$\to (c_1^2 - 4c_2)\varsigma < -4Z \cdot \varsigma < 0$

Proposition: The set of walls of type (c_1, c_2) is locally finite if $\dim(X) = 2$. Additionally if $\dim X > 2$, this is not necessarily true (multipolarization come into play).

Proposition: Let $\mathcal{C}$ be a chamber, and let $L_1, L_2 \in \mathcal{C}$ then $L_1 \overset{s}{=} L_2 \overset{s}{=} L_1 + L_2$.

proof: $L_1 \overset{s}{=} L_2$ is by definition of wall, and $L_1 + L_2$ is in the same chamber since chambers are convex and closed under re-scaling.

Corollary: Each $\mathbb{Z}$-chamber is contained in some equivalence class of $\overset{s}{=}$. Thus, an equivalence class is a union of $\mathbb{Z}$ chambers and possibly some polarization lying on walls.



A consequence of this corollary is that we can define:

(i) $\mathcal{C}_1 \overset{s}{\geq} \mathcal{C}_2$ if $L_1 \overset{s}{\geq} L_2$ for $L_i \in \mathcal{C}_i$ and $\mathcal{C}_1 \overset{s}{=} \mathcal{C}_2$ if $L_1 \overset{s}{=} L_2$

(ii) $M_{\mathcal{C}_1}(c_1, c_2) = M_{L_1}(c_1, c_2)$ and (iii) $\Delta_{\mathcal{C}_1} = \Delta_{L_1}$ and $\mathcal{E}_{\mathcal{C}_1} = \mathcal{E}_{L_1}$

Proposition: Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two chambers having a unique common face which is part of the wall W. Let $\mathcal{F}$ be the intersection $W \cap \text{Closure}(\mathcal{C}_1)$ is the common face. Assume that $Num(X) \cap \mathcal{F}$ is non empty and $L \in Num(X) \cap \mathcal{F}$. Then,

(i) $Num(X) \cap \mathcal{F}$ is contained in one equivalence class;

(ii) $\Delta_L \supset (Num(X) \cap \mathcal{C}_1) \cap (Num(X) \cap \mathcal{C}_2)$

(iii) $Num(X) \cap \mathcal{C}_1 \overset{S}{=} Num(X) \cap \mathcal{C}_2$ iff $\mathcal{E}_L \supset (Num(X) \cap \mathcal{C}_1) \cup (Num(X) \cap \mathcal{C}_2)$.

Remark: We see that E is $(Num(X) \cap \mathcal{F})$ stable iff it is both $(Num(X) \cap \mathcal{C}_1)$ and $(Num(X) \cap \mathcal{C}_2)$ stable. Thus, the study of moduli spaces of locally free rank two sheaves stable w.r.t polarizations lying on walls can be reduced to the study of sheaves stable w.r.t polarization on $\mathcal{U}$- chambers.

If one specializes to the case where X is an algebraic surface then more things can be said about the wall and chamber structure. We note that all the previous results are available but we don't have to work with the restriction that S has to be a complete intersection surface (up to scaling of L). This implies that one can work with any element of C_X.

In this context, let L_1, L_2 be non equivalent polarizations if there is a locally free rank two sheaf E such that F is L_1 stable but not L_2 stable then we have a SES: $0 \to \mathcal{O}_X(G) \to E \to \mathcal{O}_X(c_1 - G) \otimes I_Z \to 0$, where Z is locally complete intersection 0-cycle and $(2G - c_1)$ defines a non empty wall of type (c_1, c_2) with $L_1 \cdot (2G - c_1) < 0 \leq L_2 \cdot (2G - c_1)$.

Then one defines the following space:

Def: Let γ be some numerical equivalence class which defines a non empty wall of type (c_1, c_2). Let $E_\gamma(c_1, c_2)$ be the set of all locally free rank two sheaves E given by non trivial extensions:

$$0 \to \mathcal{O}_X(F) \to E \to \mathcal{O}_X(c_1 - F) \otimes I_Z \to 0$$

where F is some divisor with $(2F - c_1) \equiv \gamma'$ and Z is some locally complete intersection 0-cycle with length $\ell(Z) = c_2 + (\gamma'^2 - c_1^2)/4$.

Then one can get some decompositions of the moduli spaces such as:

Proposition: Let $\mathcal{C}$ be a chamber, and $\mathcal{F}$ be one of its faces. Then as sets,

$$\mathcal{M}_{\mathcal{C}}(c_1, c_2) = \mathcal{M}_{\mathcal{F}}(c_1, c_2) \sqcup \Big(\bigsqcup_{\gamma'} E_{\gamma'}(c_1, c_2) \Big)$$

where γ' satisfies $\gamma'.L < 0$ for some $L \in \mathcal{C}$, and runs over all numerical equivalence classes which define the wall containing $\mathcal{F}$.

This stratification is very interesting when one looks at the moduli spaces $M_N^{asd}(2, 0, c_2)$ of irreducible antiselfdual $SU(2)$-connections in a C^∞-complex VB with second chern class c_2 on N, equipped with a Riemannian metric. Donaldson proved that there exists an analytic isomorphism between $M_N^{asd}(2, 0, c_2)$ and $M_X^{\mu, lf}(2, 0, c_2)$. Results of Uhlenbeck show that the disjoint union:

$$\bigsqcup M_N^{asd}(2, 0, c_2 - \ell) \times S^\ell(N)$$

can be given a natural topology st the closure of M^{asd} in this union is a compactification of M^{asd}. One can imagine that the image of such a stratification, through Donaldson's isomorphism gives a decomposition as the one in the above proposition on $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$.