Goal: Geometric constr. of Mod. Space for ss sheaves

plan:
- Moduli Functor
- Linearization
- construct Mod. sp. as quotient of Quot scheme by grp action

Notation:
- $(X, \mathcal{O}_X(1))$ polarized proj. scheme over $k = \bar{k}$

  $\mathrm{Pic}_X / \mathrm{Pic}^0_X$

- $P \in \mathbb{Q}[z]$
- $f : S' \longrightarrow S$, $f_X := f \times \mathrm{Id}_X$.

- $\mathcal{M}' : (Sch/k)^{op} \longrightarrow Sets$

$$S \mapsto \mathcal{M}'(S) = \left\{ \begin{array}{l} \text{iso classes of } S\text{-flat families} \\ \text{of ss sheaves on } X \text{ with} \\ \text{Hilbert polynomial } P \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \text{iso classes of ss. sheaves } F \text{ on} \\ X \times S \text{ flat over } S \text{ s.t.} \\ F(s) \text{ is ss. on } \{s\} \times X \\ \forall s \in S \end{array} \right\}$$

$$\begin{array}{ccc} f : S' \longrightarrow S & \mapsto & \mathcal{M}'(f) : \mathcal{M}'(S) \longrightarrow \mathcal{M}'(S') \\ \searrow \ \swarrow & & [F] \longrightarrow [f_X^* F] \\ s & & \end{array}$$

Def: $\mathcal{C}$ cat., $M \in ob(\mathcal{C})$, $\mathcal{F}: Sch^{op} \longrightarrow Set$

- M corepresents $\mathcal{F}$ if:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\forall \phi"] \arrow[d, "\exists \alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & h_{M'} \\ h_M \arrow[ur, dotted, "\exists !"'] & \end{tikzcd} \iff \forall M' \in ob(\mathcal{C})$$

$$Mor_{\mathcal{C}}(M, M') = Mor_{Psh(\mathcal{C})}(\mathcal{F}, h_{M'})$$

$\forall M' \in ob(\mathcal{C})$

- $\alpha: \mathcal{F} \longrightarrow h_M$ universally corepresents $\mathcal{F}$ if

$$\begin{tikzcd} \mathcal{F} \arrow[r] & h_M \\ \underbrace{h_{M'} \times_{h_M} \mathcal{F}}_{\text{is corepresented by } M'.} \arrow[u] \arrow[r] & h_{M'} \arrow[u, "\phi"'] \end{tikzcd}$$

- if $F \in \mathcal{M}'(S)$, $\forall L \in Pic(S)$,

$$F_s \cong (F \otimes p^* L)_s, \quad p: X \times S \xrightarrow[p_2]{} S$$

$\rightsquigarrow \forall F, F' \in \mathcal{M}'(S)$, $F \sim F'$ iff $F' = F \otimes p^* L$ for some $L \in Pic(S)$

Rem:
- M represents $\mathcal{F} \Longrightarrow M$ universally corep. $\mathcal{F}$.
- M corepres. $\mathcal{F} \Longrightarrow M$ is unique up. to. iso

Def: A scheme M is a <u>Moduli space</u> of ss sheaves if it corepresents $\mathcal{M} := \mathcal{M}'/\sim$

Def: • G alg. k-group

$(G, \mu : G\times G \to G, e : \operatorname{Spec} k \to G, i : G \to G)$

Sps. $G \circlearrowright X$ via $\sigma : G \times X \to X$.

• $\forall x \in X$,

$G.x$ (the orbit) = image $\left(\begin{array}{l}\sigma_x : G(k) \to X(k) \\ \quad g \to \sigma(g,x)\end{array}\right)$

• the stabilizer G_x is the fiber prod of:

$\sigma_x : G \to X$ and $\operatorname{Spec} k \to X$.

E.g: $\mathbb{G}_m := \operatorname{Spec} k[t, t^{-1}] \circlearrowright \mathbb{A}^m$ by

$t.(a_1, \dots, a_m) = (ta_1, \dots, ta_m)$

- origin
- punctured lines through origin.

$$\mathbb{P}^{m-1} = \mathbb{A}^m \setminus \{0\} \big/ \mathbb{G}_m$$

Def: a categorical quotient for σ is a k-scheme Y corepresenting $X/G : (Sch/_k)^{op} \longrightarrow Sets$

$$T \longrightarrow X(T)/G(T)$$

Y corep. $X/G \rightsquigarrow G/_X(X) \longrightarrow h_Y(X)$

$$[Id_X] \mapsto (Y \xrightarrow{\pi} Y)$$

$$\begin{tikzcd} X \arrow[r, "\forall \phi"] \arrow[d, "\exists \varphi"'] \arrow[dr, phantom, "\circ"] & Z \\ Y \arrow[ur, dotted, "\exists ! \psi"'] & \end{tikzcd}$$

φ, ϕ are G-invariant

E.g: $\mathbb{A}^n \longrightarrow \operatorname{Spec} k$ is cat. quotient for

$$\mathbb{G}_m \circlearrowright \mathbb{A}^n$$

Def: G aff. alg. $k$group acting on k-scheme X

- $\varphi : X \longrightarrow Y$ is good quotient if
  - ☆ φ affine + surj + invariant
  - ☆ $U \overset{open}{\subset} Y \iff \varphi^{-1}(U) \overset{open}{\subset} X$

* $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ iso

* if W is closed subset of X that is G-invariant, then $\varphi(W)$ is closed subset of Y

  if W_1, W_2 invariant disjoint closed then, $\varphi(W_1) \cap \varphi(W_2) = \phi$.

- Moreover, $\varphi : X \to Y$ is called geometric quotient, if the fiber over each point in Y is a single orbit.

<u>Rem</u>: • good quotient $\Longrightarrow$ universal quotient (categorical)

• if $\varphi : X \to Y$ is good quotient, if X is red., irred, integr, normal, then Y is the same.

- $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ is a good quotient
$(x,y) \to x \cdot y$

for $\mathbb{G}_m \circlearrowright \mathbb{A}^2$ via $t.(x,y) \longmapsto (tx, t^{-1}y)$

Def: alg. k-group $G \circlearrowright X$ k-scheme of finite type via $\sigma : G \times X \longrightarrow X$

A <u>G-linearization</u> of $F \in Coh(X)$ is an iso. $\phi : \sigma^* F \longrightarrow p_X^* F$

w/ $p_X : X \times G \longrightarrow X$ (proj) and we have the following cocycle condition:

$$(\mu \times Id_X)^* \phi = p_{23}^* \phi \circ (Id_G \times \sigma)^* \phi$$

$p_{23} : G \times G \times X \longrightarrow G \times X$ proj.

- morph. of G-linearized q-coh sheaves on X is a morph. $f : F \longrightarrow F'$ commuting with G-linearizations:

$$\begin{tikzcd} \sigma^* F \arrow[r, "b"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \sigma^* F' \arrow[d, "\phi'"] \\ p_X^* b \arrow[r, "p_X^* b"'] & p_X^* b \end{tikzcd}$$

<u>E.g.:</u> Aff. alg. group $G \circlearrowleft X$ k-scheme.

$\chi : G \longrightarrow \mathbb{G}_m$ character.

$\chi \rightsquigarrow$ linearisation on the trivial bdl

$\mathbb{A}^1 \times X \longrightarrow X$ by $g.(x, z) = (g.x, \chi(g).z)$

$\underset{\text{reductive groups}}{\overset{\text{restrict to}}{\rightsquigarrow}}$

<u>Th:</u> reductive $G \circlearrowleft X$ aff. sch. of finite type

$A(X)$, $Y = \mathrm{Spec}(A(X)^G)$

then $A(X)^G$ is f.g. over k, so that

Y is o.f.t over k, $\pi : X \to Y$ is

a universal good quotient for the action of G.

X proj. k-scheme.

Def. • $x \in X$, x is semistable w.r.t. G-linearized ample line bdle L, if $\exists m \in \mathbb{Z}$, $\exists s \in H^0(X, L^{\otimes m})^G$ s.t. $s(x) \neq 0$.

• $x \in X$, x is stable if moreover G_x is finite and $G.x$ is closed in the open set of ss points in X.

• x is properly stable if it's semistable but not stable.

• $X^s(L)$ (resp. $X^{ss}(L)$) are open G-inv. subsets of X.

[Mumford]

Thm Let G reductive / $\mathbb{C}$, X ~~proj.~~ q-proj. scheme w/ G-linearized ~~ample~~ line bdl L

then $\exists$ ~~proj~~ q-proj scheme Y, $\exists\ \pi : X^{ss}(L) \to Y$ universal good quotient.

Moreover, $\exists\ Y^s \underset{\text{open}}{\subset} Y$ s.t. $X^s(L) = \pi^{-1}(Y^s)$

and $\varphi : X^s(L) \longrightarrow Y^s$ is a geometric quotient.

Proof sketch $Y := \operatorname{proj} R^G$

$$R^G := \Big(\bigoplus_{n \geq 0} H^0(X, L^{\otimes n}) \Big)^G$$

L is ample $\Longrightarrow \forall\ s \in R^G_+ ,\ X_s = \{x \in X, s(x) \neq 0\}$ is affine

$\rightsquigarrow$ get good GIT quotients by glueing affine GIT quotients.

$\boxed{/\!/}$

$\lambda : \mathbb{G}_m \longrightarrow G$ non triv. 1-PS

$G \curvearrowright X \rightsquigarrow \mathbb{G}_m \curvearrowright X$

X proj $\Longrightarrow$ $\mathbb{G}_m \longrightarrow X \underset{\text{uniquely}}{\overset{\text{extends}}{\rightsquigarrow}} f : \mathbb{A}^1 \longrightarrow X$

$$\begin{tikzcd}
\mathbb{A}^1 \setminus \{0\} \arrow[r, "\lambda"] \arrow[d, hook] & G \arrow[d] & g \arrow[d, mapsto] \\
\mathbb{A}^1 \arrow[r, "f"'] & X & \sigma(x, g)
\end{tikzcd}$$

$f(0) = \lim_{t \to 0} \sigma(x, \lambda(t)) \rightsquigarrow \mathbb{G}_m$ acts on $L_{f(0)}$

with a weigh r.

$\rightsquigarrow$ if ϕ : G-lin. of L, then

$$\phi(f(0), \lambda(t)) = t^r \, id_{L_{f(0)}}$$

Define: $\mu^L(x, \lambda) := -r$

<u>Theorem</u>: <u>Hilbert-Mumford criterion</u>: $x \in X$.

- $x \in X^{ss}(L)$ iff $\mu^L(x, \lambda) \geq 0$, $\forall \lambda$ 1PS of G
- $x \in X^s(L)$ —— —— > 0 ——

<u>Def:</u> $m \in \mathbb{Z}$, $\mathcal{F} \in \mathrm{coh}(X)$ is said m-regular

if $H^i(X, \mathcal{F}(m-i)) = 0 \quad \forall i > 0.$

$\xrightarrow{\text{we know}}$ family of ss sheaves on X with fixed Hilbert poly. is bounded.

$\Longrightarrow \exists m \in \mathbb{Z}$ s.t. $\forall F \in ss_P(X)$ is m-reg.

$\Longrightarrow F(m)$ is globally generated $\quad P(m) = h^0(F(m))$

Set $V := k^{\oplus P(m)}$, $\quad \mathcal{H} := V \otimes \mathcal{O}_X(-m)$

then:

$$\begin{tikzcd} V \arrow[r, "\cong"] & H^0(F(m)) \arrow[r, squiggly, "\text{surj}"] & \mathcal{H} \arrow[r, "\cong"] & H^0(F(m)) \otimes \mathcal{O}_X(-m) \arrow[d, "ev"] \\ & & & F \end{tikzcd}$$

we get $[\rho : \mathcal{H} \to F] \in \mathrm{Quot}(\mathcal{H}, P)$ representing

$$(\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$$

$$T \longmapsto \left\{ \begin{array}{l} T\text{-flat coh. quotients} \\ \mathcal{H} \otimes \mathcal{O}_T \longrightarrow F \text{ w/} \\ \text{Hilbert polynomial } P \end{array} \right\}$$

$$[e] \subset R \underset{\text{open}}{\subset} \mathrm{Quot}(\mathcal{H}, P)$$

$$R := \left\{ [\mathcal{H} \to E] \;\middle|\; \begin{array}{l} E \text{ is ss and} \\ V \cong H^0(E(m)) \end{array} \right\}$$

$$GL(V) \circlearrowright \mathrm{Quot}(\mathcal{H}, P)$$

$$Z(GL(V)) \subset \bigcap_{[e] \in \mathrm{Quot}(\mathcal{H}, P)} GL(V)_{[e]}$$

$$PGL(V) \circlearrowright \mathrm{Quot}(\mathcal{H}, P) \rightsquigarrow SL(V) \circlearrowright \mathrm{Quot}(\mathcal{H}, P)$$

Plucker embedding + Grothendieck + Some extra arguments

$$\rightsquigarrow L_\ell := \det\left(p_*\left(\tilde{F} \otimes q^* \mathcal{O}_X(\ell)\right)\right),$$

is very ample + $GL(V)$ - linearized.

<u>Th:</u> Fix $m \gg 0$, $\ell \gg 0$ Then

$R = \overline{R}^{ss}(L_\ell)$ and $R^s = \overline{R}^s(L_\ell)$. $\boxtimes$

<u>Proof idea</u>: ① compute weights of certain action

of $\mathbb{G}_m$

(2) determin (Semi) stability condition (GIT) of : $[\rho: V \otimes \mathcal{O}_X(-m) \to F] \in \overline{R}$

using HM crit. in terms of nbr of glob. sec. of $F' \subset F$.

(3) relate it to stability of F.

(3) is realized via $\boxed{\text{Le Potier theorem}}$. ▨

Theorem [Le Potier]:

Let $S_m(P)$ denote the iso classes of pure sheaves with fixed Hilbert Poly. P, and satisfying:

i) $P(m) \leq h^0(F(m))$

ii) $\forall F' \subset F$, $F' \neq 0$ and F' coherent,

$$\frac{h^0(F(m))}{r'} \leq \frac{h^0(F(m))}{r}$$

Then $\exists$ a s.t. $\forall m \geq a$, $S_m(P) = \left\{ \begin{array}{l} \text{iso classes of ss} \\ \text{sheaves of Hilbert} \\ \text{poly. } P \end{array} \right\}$ ▨