Hochschild cohomology of algebras.

Take an associative algebra $\Lambda (= kQ/I)$ and consider it as Λ-Λ bimodule over itself, or equivalently as a right $\Lambda^{op} \otimes \Lambda$ module so we introduce $\Lambda^{op} \otimes \Lambda$ as the envelopping algebra Λ^e of Λ.

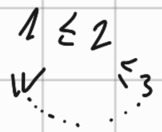
Take M a right Λ^e mod. then form

$$0 \leftarrow M \leftarrow M \otimes \Lambda \leftleftarrows M \otimes \Lambda^{\otimes 2} \;\overset{\leftarrow}{\leftleftarrows}\; M \otimes \Lambda^{\otimes 3} \;\leftleftarrows\!\leftleftarrows \;\dots$$

$$m\lambda \leftarrow m \otimes \lambda$$

s.t. $\forall i \in [\![0, n-1]\!]$, $\partial_i = id^{\otimes i} \otimes \mu \otimes id^{\otimes n-i-1}$

$\partial_n = \mu \otimes id^{n-1} \circ (1, \dots, n+1)$ (take a cyclic multiplication μ



$\rightsquigarrow$ a differential $\sum_{n \geq 0} \sum_{k=0}^{n} (-1)^n \partial_k := d$

$\rightsquigarrow$ a complex $(C(\Lambda, M), d)$ and call the homology of this Hochschild homology denoted $HH_\bullet(\Lambda, M)$

Now look at $0 \to M \xrightarrow{d} Hom(\Lambda, M) \xrightarrow{d} Hom(\Lambda^{\otimes 2}, M) \xrightarrow{d} \dots$

$$\longrightarrow \mathrm{Hom}(\Lambda^{\otimes 3}, M) \longrightarrow \cdots$$

$$\mathrm{Hom}(\Lambda^{\otimes m}, M) \longrightarrow \mathrm{Hom}(\Lambda^{\otimes m+1}, M)$$

$$= \left(\mathrm{id}^{\otimes i-1} \otimes \mu \otimes \mathrm{id}^{\otimes m-i} \right)^{\#}$$

$\#$ ← adj. w.r.t. tensor Hom adjunction

with

$$\partial_0 = \left(\mu \circ \mathrm{id}_\Lambda \otimes \mathrm{id}_{\mathrm{Hom}(\Lambda^{\otimes m}, M)} \right)^{\#}$$

$$\partial_{m+1} = \left(\mu \circ \mathrm{id}_{\mathrm{Hom}(\Lambda^{\otimes m}, M)} \otimes \mathrm{id}_\Lambda \right)^{\#}$$

$$df(\lambda_1, \dots, \lambda_{m+1}) = \sum_{k=1}^{m+1} (-1)^k f(\lambda_1 \otimes \dots \otimes \lambda_{k-1} \lambda_k \otimes \dots \otimes \lambda_{m+1})$$

$$+ \lambda_1 f(\lambda_2 \otimes \dots \otimes \lambda_{m+1})$$

$$+ (-1)^{m+2} f(\lambda_1 \otimes \dots \otimes \lambda_m) \lambda_{m+1}$$

And the coho of this complex denoted $HH^\circ(\Lambda, M)$ is the Hochschild coho.

<u>interpretation:</u> • $HH^0(\Lambda)$ is the Center of Λ.

• $HH^1(\Lambda)$ is the outer derivations

• $HH^2(\Lambda)$ are Hochschild extensions. ($0 \to \Lambda \to$ $\mathrm{Hom}(\Lambda,\Lambda) \to \mathrm{Hom}(\Lambda^{\otimes 2},\Lambda)$

<u>Lemma</u> $HH^\bullet(\Lambda, M) \cong \mathrm{Ext}^\bullet_{\Lambda^e}(\Lambda, M)$

$HH_\bullet(\Lambda, M) \cong \mathrm{Tor}_\bullet^{\Lambda^e}(\Lambda, M)$

<u>Def:</u> The bar resolution of Λ is the projective Λ^e-module resolution of Λ is the projective Λ^e-module resol. of Λ.

$$\cdots \xrightarrow{\beta} \Lambda^{\otimes 3} \xrightarrow{\beta} \Lambda^{\otimes 2} \xrightarrow{\beta} \Lambda \longrightarrow 0$$

$$\beta = \sum_{i=0}^{m} (-1)^i \, \mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes m-i-1}$$

To compute $\mathrm{Hom}(\Lambda^{\otimes m}, M) \xrightarrow{d} \mathrm{Hom}(\Lambda^{\otimes m+1}, M)$

$\rightsquigarrow$ we find that $d = \sum_{i=0}^{m+1} (-1)^{i+1} \partial_i$