Spectral covers (Kumi)

__Recall:__ BNR:

If (E,ϕ) s.t. $h(E,\phi) = a = (a_1, \dots, a_r)$

$$\begin{tikzcd} \pi^* K \arrow[r] \arrow[d] & K \arrow[d] \\ X_a \subset K \arrow[r] \arrow[u, bend left, "\eta"] & X \end{tikzcd}$$

$X_a \subset K \hookrightarrow \pi^{\otimes r} + \pi^* a_1 r^{\otimes r-1} + \dots + \pi^* a_r$ $\in \pi$

$h^{-1}(a) \cong Pic(X_a)$

__Th 4.1__: $pol_W : t^d /\!/ W \to A \simeq \prod S^{l_i}/A$

$$t^d /\!/ W \to B \subset A, \quad B = \mathrm{Chow}\, A^d$$

B ← closed subscheme of A

__plan:__ I) $d=1$ II) $d \geq 2$ III) Gen of BNR to $B_X^{\heartsuit}$

I) if $G = GL_n$, $t = \mathbb{A}^n = (x_1, \dots, x_n)$

spect curve of C $\rightsquigarrow C^\circ \to C$, consider $S_{n-1} \curvearrowright \mathbb{A}^n$ fixing x_n

$C^\circ := \mathbb{A}^n / S_{n-1} \longrightarrow \mathbb{A}^n / S_n$

$[c] \longrightarrow [c]$

$\rightsquigarrow$ $C : C^\circ \xrightarrow{\alpha} C \times \mathbb{A}^1 \xrightarrow{proj_1} C$

$(c'_1, \dots, c'_{n-1}, x_n) \overset{\alpha}{\mapsto} (c_1, \dots, c_n, t)$

$c_1 = c'_1 + x_n$, $c_{n-1} =$

$$\xrightarrow{\text{Sym}_m} (c_1, \dots, c_m)$$

$$c_1 = x_1 + \dots + x_m, \qquad c_m = x_1 \cdots x_m$$

The fibers of $C^\bullet \longrightarrow C$ are just $c \in C$ s.t. $c = c_1, \dots, c_m$

$$C \times \mathbb{A}^1 \cap \iota(C^\bullet) \longrightarrow C$$

$$p^{-1}(c) = \{ t \in \mathbb{A}^1 \mid t^m - c_1 t^{m-1} + \dots + (-1)^m c_m = 0 \}$$

II) for $G = GL_m$, $\mathfrak{t}^d = (\mathbb{A}^d)^m$ then

$$\mathfrak{t}^d // W \simeq \mathrm{Chow}(\mathbb{A}^d) \simeq (\mathbb{A}^d)^m // S^m$$

$$x = [x_1, \dots, x_m]$$

$\rightsquigarrow$ consider $\chi_{\mathbb{A}^d} : \mathrm{Chow}\,\mathbb{A}^d \times \mathbb{A}^d \longrightarrow \underbrace{S^m \mathbb{A}^d}_{\text{v.s.}}$

$$[x_1, \dots, x_m]\, x \mapsto (x - x_1) \cdots (x - x_m)$$

$$\chi_{\mathbb{A}^d}\left([x_1,\dots,x_m], x\right) = x^m - c_1 x^{m-1} + \dots + (-1)^m c_m \quad \text{where}$$

x_1 is a sym. poly on $(x_1, \dots, x_m)$

<u>Def.</u>: $\mathrm{Cayley}_m(\mathbb{A}^d) = \chi^{-1}_{\mathbb{A}^d}(0)$.

<u>prop</u>: $\mathrm{Cayley}_m(\mathbb{A}^d) \overset{\chi}{\hookrightarrow} \mathrm{Chow}(\mathbb{A}^d) \times \mathbb{A}^d \xrightarrow[pr_1]{} \mathrm{Chow}_m(\mathbb{A}^d)$

1) $\rho := pr_1 \circ \chi : \mathrm{Cayley} \longrightarrow \mathrm{chow}$. is a functor which is étale over $\mathrm{Chow}^0_m(\mathbb{A}^d)$ (multiplicity-free)

2) For every $x = [x_1^{m_1} \dots x_m^{m_m}]$, $\rho^{-1}(a) = \mathrm{Cayley}_m(a)$

$= \coprod \mathrm{Spec}\left(\mathcal{O}_{\mathbb{A}^d, x_1} / m^{m}_{x^n}\right)$ fat bdle

3) Let F be a finite $\mathcal{O}_{\mathbb{A}^d}$-mod of length m, $a \in \mathrm{Chow}$ be its spectral data then F is supported by cayley.

(this is a gen. of Cayley-Hamilton theorem)

<u>Pf</u>: generators of Cayley $\rightsquigarrow v \in V^d$ lin. forms on $\mathbb{A}^d$.

$$[v] : \mathrm{Chow}_m(\mathbb{A}^d) \longrightarrow \mathrm{Chow}_m(\mathbb{A}^1)$$
$$[x_1, \dots, x_m] \longmapsto [v(x_1), \dots, v(x_m)]$$

$$\rightsquigarrow \begin{tikzcd} \mathrm{Chow}_m \mathbb{A}^d \times \mathbb{A}^d \arrow[r, "\chi_{\mathbb{A}^d}"] \arrow[d, "{[v] \times v}"'] & S^m \mathbb{A}^d \arrow[d, "S^m(v)"] \\ \mathrm{Chow}_m \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "\chi_{\mathbb{A}^1}"'] & S\mathbb{A}^1 \cong \mathbb{A}^1 \end{tikzcd} \quad \rightsquigarrow b_v = \chi_{\mathbb{A}^1} \circ ([v] \times v)$$

vanishes on Cayley

of deg m.

$$\rightsquigarrow b_v(a, x) = (v(x) - v(x_1)) (v(x) - v(x_n)) \qquad (*)$$

$\rightsquigarrow$ generates the ideal of Cayley as v varies in V^d

(1) Let $v_1, \dots, v_d$ be a basis of v_d.

Let $Z(b_{v_1}, \dots, b_{v_d}) \subset \mathrm{Chow}_m \mathbb{A}^d \times \mathbb{A}^d$ is finite of degree m over $\mathrm{Chow}\, \mathbb{A}^d$.

$\rightsquigarrow$ Cayley is finite over Chow. (properness is missing)

2) Let $a = [x_1^{m_1} \dots x_m^{m_m}]$

$\mathrm{Cayley}_m(a)$ cut $m_{x_1}^{m_1} \dots m_{x_m}^{m_m}$

$\hookrightarrow$ is cut $I_a = I_{x_1} \dots I_{x_m}$

where A/I_{x_1} is supported by some thickening by x_i

$\rightsquigarrow$ $x \notin \{x_1, \ldots, x_m\}$, find $f \in I_a$ but $f \notin m_{x_1}$

(*) choose a $v \in V$ s.t. $v(x) \neq v(x_1)$

$f_v(x) \neq 0 \longrightarrow$

focus on x_1, we just need to prove that the image of

$f_v(a)$ in $S(V_d)_{x_1}$ generate the ideal m_{x_1}

$\xrightarrow[\text{Nakayama}]{}$ NTP that images of $f_v(a)$ in $m_{x_1}^{m_1}/m_{x_1}^{m_1+1}$

generate it.

Observe that, if $v \in V_d$ s.t. $v(x_1) \neq v(x_i)$

then $(v(x_1) - v(x_2)), \ldots, (v(x) - v(x_m))$ are invertible

[illegible]. It's enough for $v \in V_d$ s.t.

$v(x_1) \neq v(x_i)$ the family $(v(u) - v(x_i))$ generate

$m_{x_1}^{m_1}/m_{x_m}^{m_1+1}$. Take the image $m_{x_1}/m_{x_1}^2 \xrightarrow{m^{th}} m_{x_1}^{m_1}/m_{x_1}^{m_1+1}$

3) by CRT : If a is finite $S(V_d)$-module of length n then it's supported by finite thickening.

pf: 3): Since f is supported by a thickening $\rightsquigarrow$ $S(V_d)_x$-mod struct.

consider $F \subset m_x F \subset m_x^2 F \subset \dots$

$\underset{\text{Nakayama}}{\rightsquigarrow}$ As long as $m_x^d F \neq 0$

then $\dim x_1^{m_i} F / x_1^{m_i+1} F \geq 1$ for $i = 0, \dots, m$

So $m+1 \leq n$, we have $m_x F = 0$.

as in $d = 1$, $S_{n-1} \curvearrowright (\mathbb{A}^d)^n$ fixing the n^{th} of $\mathbb{A}^d$

$\rightsquigarrow (\mathbb{A}^d)^n // S_{n-1} \longrightarrow (\mathbb{A}^d)^n / S_n$

$$\begin{array}{ccc} L: (\mathbb{A}^d)^n //_{S_{n-1}} \times \mathbb{A}^d & \longrightarrow & \mathrm{Chow}_n(\mathbb{A}^d) \times \mathbb{A}^d \\ \| & & \cup \\ \mathrm{Chow}_{n-1}(\mathbb{A}^d) & & \mathrm{Cayley} \end{array}$$

$[x_1, \dots, x_{n-1}], x_n \longmapsto [x_1, \dots, x_n], x_n$

Remark: [Drinfeld] is f an iso ?

($\Longleftrightarrow$ Cayley reduced + normal)

$\rightsquigarrow$ build spect. cover

$\forall\, b \in B_X(k)$, we have $b : X \to [B/GL_d]$

over the cat. of X. thus:

Spec. cover of X

$$\begin{tikzcd} X_b^\circ \arrow[d, "\pi_b"] & {[B^\circ/GL_d]} \arrow[d] \\ X \arrow[r, "b"'] & {[B/GL_d]} \end{tikzcd}$$

π_b is a finite cover because $B^\circ \longrightarrow B$ is finite.

Moreover, if $b \in B_X^\heartsuit(k)$, i.e. $b(X)$ has non ϕ intersection with $[B^\circ/GL_d]$.

$\rightsquigarrow$ π_b is generically finite étale of deg n.

III) Generalise BNR

df: we say that $M \in \mathrm{Coh}$ over a f.t. scheme Y is

Cohen-Macaulay of cod d iff

1) $\operatorname{cod}(\operatorname{Supp} M) = d$

2) $H^i(D(M)) = 0$, $\forall i \neq d$.

$\rightsquigarrow$ CM :: let R be finite R-alg deg m.

with A a reg ring of f.t dim m

M R-mod is loc. free of rank n iff M is CM

Prop 6.3: $\forall b \in B^{\heartsuit}(h)$, the fiber $h^{-1}(b)$ is

$\cong$ to the stack of max CM of gen. rk 1 over X_b