Lecture: Spectral + cameral cover

$\mathrm{Pic}_0(\mathcal{C}_a) \longleftrightarrow h^{-1}(a) \in \mathcal{M}$ when $\mathcal{C}_a$ is regular.

The fiber is an abelian variety + integrable system.

(V, φ), $\varphi \in \Gamma(\mathrm{End}(V) \otimes K)$

E a G-principal bdl.

$$\begin{array}{l} E \\ \downarrow \quad \rho : G \to GL(V) \quad \rightsquigarrow \text{recover } V \\ X \end{array}$$

$\mathfrak{t} \in \mathfrak{g} := \mathrm{Lie}(G)$

↳ Cartan subalg.

$$\mathbb{C}[\mathfrak{t}]^W \longleftrightarrow \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$$

$\mathfrak{g} \to \mathfrak{t}/W$ (apply Spec)

$$\begin{tikzcd} \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t} \arrow[r] \arrow[d] & \mathfrak{t} \arrow[d] \\ \mathfrak{g} \arrow[r] & \mathfrak{t}/W \end{tikzcd}$$

W-cover

$$
\begin{tikzcd}
\tilde{X} = \varphi^*(\arrow[d] &) \arrow[r] & \mathrm{end}(E) \arrow[d] & \arrow[r] & \mathfrak{t}\otimes K \arrow[d] \\
X \arrow[rr, "\varphi"'] & & \mathrm{ad}(E)\otimes K \arrow[rr] & & \mathfrak{t}\otimes K/W
\end{tikzcd}
$$

$\rho = \oplus \rho_i \quad , \quad \tilde{X}_\rho = \cup \tilde{X}_{\rho_i}$

$$V = \bigoplus_\lambda V_\lambda = \bigoplus_{\lambda\in D} \bigoplus_{\mu \in W_\lambda} V_\mu$$

define $\tilde{X}_\lambda$ for each $\lambda \in D$, $\tilde{x} = \sqcup_{m_\lambda} \tilde{x}_\lambda$ w/ $m_\lambda = \dim V_\lambda$

$\overline{P}_\lambda : \mathfrak{t} \to \mathbb{C}[x] \rightsquigarrow P_\lambda : \mathfrak{g} \to \mathbb{C}[x]$

$\rightsquigarrow \prod_{\mu \in W_\lambda} (x - \mu) \qquad P_\lambda(\varphi) : K \to K^n$

$\hat{X}_\lambda = \left(P_\lambda(\varphi)\right)^{-1}(0)$: cameral cover of X

$\tilde{X} \xrightarrow{J_\lambda} \tilde{X}_\lambda$ locally given by $\tilde{\mathfrak{g}} \to \mathfrak{g} \times \mathbb{C}$

$(g, z) \mapsto (g, \lambda(t))$

If $\mu_1, \mu_2 \in W_\lambda$ take same value on $\tau \in \mathfrak{t}$ ⇝ singularities.

If P a parabolic subgrp, $\lambda \in \mathcal{C}_P$ then $\tilde{X}/W_P \to \tilde{X}_\lambda$

$$W_P = N \cap P / T$$

Cameral covers are W covers for $G = GL(n)$, $W = S_n$.

$T \subseteq G$ a max torus, $N_G(T) =: N$,

$x \in \mathrm{Lie}(G)$ is regular if $Z_{\mathfrak{g}}(x)$ has minimal dimension.

$\mathfrak{a} \subset \mathfrak{g}$ Lie subalg. if regular centraliser of $a = Z_{\mathfrak{g}}(x)$

for some x reg. elem.

G/N ⇝ paramet cartan subalg $\hookrightarrow \overline{G/N}$ = sp. param. reg centralisers

$$\overline{G/T} = \{(a, b) \; a \in \overline{G/N}, \; b \text{ Borel subalg } a \subset b\}$$

$$\begin{array}{c} \overline{G/T} \\ \downarrow \\ \overline{G/N} \end{array}$$

X is a scheme, a family of cartan subalg $/ X$

is a morph $X \to G/N \longleftrightarrow G$-equiv map

$$X \times G \to G/N$$

Higgs bdl is a pair (E, σ) , E principal bdl,
σ a G-equiv map $E \to \overline{G/N}$

$\mathrm{Higgs}(X)$ = cat of Higgs bdles

Def: a W-cover of a scheme X is a scheme $\hat{X} \xrightarrow{\pi} X$ finite + flat
s.t. locally $\pi_* \mathcal{O}_{\hat{X}} \cong \mathcal{O}_X \otimes \mathbb{C}[W]$ as $\mathcal{O}_X$-mods w/ W-action

Def: a Cameral cover is a W-cover $\hat{X}$ s.t. locally it looks
like pullback of the W-cover $\mathfrak{t} \to \mathfrak{t}/W$

Def: $\mathrm{Cam}(X)$: obj: cameral covers $/X$
morph: W-equiv. isos.

Th: $\exists$ natural functor : $F: \mathrm{Higgs}(X) \to \mathrm{Cam}(X)$
(E, σ)

$$\begin{tikzcd}
\sigma^*(\overline{G/T}) \arrow[r] \arrow[d] & \overline{G/T} \arrow[d] \\
E \arrow[r, "\sigma"'] & \overline{G/N}
\end{tikzcd}$$

since $\sigma^*(\overline{G/T}) \to E$ is G-equiv
$\Rightarrow$ it is the pullback of a cam cov.

$\mathrm{Higgs}_X(\tilde{X})$ = fiber of $\tilde{X}$

Objects : (E, σ, τ) : $(E, \sigma) \in \mathrm{Higgs}(X)$ $F: (E, \sigma) \xrightarrow[\sim]{\tau} \tilde{X}$

morphs : $(E_1, \sigma_1, \tau_1) \longrightarrow (E_2, \sigma_2, \tau_2)$ are the morphs

in $\mathrm{Hom}((E_1, \sigma_1), (E_2, \sigma_2))$ s.t.

$$\tilde{X} \xrightarrow{\tau_1^{-1}} F(E_1, \sigma_1) \xrightarrow{F(\lambda)} F((E_2, \sigma_2)) \xrightarrow{\tau_2} \tilde{X}$$

<u>Def</u>: sheaf of cats ...

Then How to see $\mathrm{Higgs}_{\tilde{X}}$ as a sheaf of cats.

$$\mathrm{Higgs}_{\tilde{X}}(U) := \mathrm{Higgs}_{\hat{U}}(U)$$

<u>Theo</u> : $\mathrm{Higgs}_{\tilde{X}}$ is a gerb over $\mathrm{Tors}_{T\tilde{X}}$.

<u>Def</u> A <u>picard cat.</u> is a groupoid with str. of Tensor alg. s.t.

all obj are invertible

A Sheaf of Pic cat is a sheaf cat P s.t. $P(U)$ is

a picard cat $\forall U$ and f^* are compat. w/ $\otimes$.

<u>Def</u>: A gerb over a Picard cat $\mathcal{P}$ is a cat. Q s.t. $\forall C \in Q$

$\mathcal{P} \longrightarrow Q$

$P \to \mathrm{Aut}(P, C) \in Q$

Def: A gerb over a sheaf of Picard cat $Q(U)$ is a gerb over $\mathcal{P}(U)$

$\exists\, U \to X$ s.t. $Q(U)$ non empty.

$0 \to \mathcal{A}' \to \mathcal{A} \hookrightarrow \mathcal{A}'' \to 0$ a SES of sheaves of ab. groups.

$\tau_{\mathcal{A}''}$ is an $\mathcal{A}''$-torsor

Define $Q(U)$: all possible liftings of $\tau_{\mathcal{A}''|U}$ to be an $\mathcal{A}$-torsor.

Q is a gerb over $\mathrm{Tors}_{\mathcal{A}'}$

Th: $\exists$ bij btw iso classes of $\mathrm{Tors}_{\mathcal{A}}$-gerbs and $H^2(X, \mathcal{A})$

Moreover, $Q(X) \neq 0$ iff $0 \in H^2(X, \mathcal{A})$

$$\hat{T}_{\tilde{X}}(U) := \underset{W\text{-equiv}}{\mathrm{Hom}}(\tilde{U}, T)$$

if α is a root $\rightsquigarrow S_\alpha$

if $f: \tilde{U} \to T$ $\quad \alpha \circ f : \tilde{U} \to \mathbb{C}^*$

$$s_\alpha(\alpha \circ f)\big|_{D_U^\alpha} \; D^\alpha_{\tilde{U}} \longrightarrow \{\pm 1\}$$

$$T_{\tilde{X}}(U) = \{ f \in \hat{T}_{\tilde{X}}(U) : s_\alpha(\alpha \circ f)|_{D^\alpha} = 1, \ \forall \alpha \}$$

Th: $\mathrm{Higgs}_{\tilde{X}}$ is equivalent to the sheafification of the cat of the category of R-twisted, N-shifted, W-equiv T-bdl over $\tilde{X}$.

W acts on $\tilde{X}$, W acts on T via conjugation.
$\quad\quad\quad\quad\quad\quad\quad$ " N/T

If $\mathcal{L}$ is a T-bundle $/\tilde{X}$: $w^*(\mathcal{L})$: the T-bdl obtained using both actions
$w \in W$

Def: $\mathcal{L} \to \tilde{X}$ a T-bdl is weakly W-equiv if $w^*(\mathcal{L}) \cong \mathcal{L}$ $\forall w \in W$

Def: α a root

$$R^\alpha_X \to \mathcal{O}(D^\alpha_{\tilde{X}}) \text{ if } \tilde{X} \text{ is integral}$$

$$R^\alpha_X \to I^*_X \text{ the cat sheaf of symbols } \{g\} \text{ s.t.}$$

$\{g \in \mathcal{O}_{\tilde{X}} \; s(g) = -g\} \rightsquigarrow R^\alpha_X$ is its inverse.

$\alpha^\vee : \mathbb{C}^* \to T$, $\mathcal{R}^\alpha_X = \alpha^\vee(R^\alpha_X)$.

$w \in W$, $\mathcal{R}^w_X = \bigotimes_{\alpha \in J} \mathcal{R}^\alpha_X$, $J = \{$positive roots α s.t. $w(\alpha)$ is negative$\}$

$$\mathcal{R}^{w_1 w_2}_X \simeq w_2^*(\mathcal{R}^{w_1}_X) \otimes \mathcal{R}^{w_2}_X$$

Def: a T-bdl $\mathcal{X}$ over $\tilde{X}$ is weakly $\mathcal{R}$-twisted W-equiv

if $\forall w \in W$, $w^*(\mathcal{X}) \otimes \mathcal{R}^w_x \simeq \mathcal{X}$

For a weakly w-equiv $\mathcal{X}$, define $Aut(\mathcal{X})$

to be $\{(w, \varphi) : w \in W, \varphi : w^*(\mathcal{X}) \to \mathcal{X}\}$

$$0 \to Hom(\tilde{X}, T) \to Aut(\mathcal{X}) \to W \to 0$$

For a weakly $\mathcal{R}$-twisted W-equiv T-bdl

$$Aut_R(\mathcal{X}) = \{(w, \varphi) : w \in W, \varphi : w^*(\mathcal{X}) \otimes \mathcal{R}^w \to \mathcal{X}\}$$

$$0 \to Hom(\tilde{X}, T) \to Aut_R(\mathcal{X}) \to W \to 0.$$

$Higg'(X) :=$ cat of R-twisted, N-shifted, W-equiv T-bdl / $\tilde{X}$.

- a weakly R-twisted W-equiv T-bdl over $\tilde{X}$.

-
$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\gamma"] & W \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & Hom(\tilde{X}, T) \arrow[r] & Aut_R(\mathcal{X}) \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

, $\forall \alpha_i$ root, $\forall m_i \in N_i$ we have $\beta(m_i) = \alpha_i(\mathcal{X})|_{D^{\alpha_i}} \to R^{\alpha_i}_{d}|_{D^{\alpha_i}}$
+ some compat. conds

$\alpha_i \rightsquigarrow M_i$ corresponding minimal Levi subgrp

$N \cap [M_i, \pi_i] : \pi \to \mathcal{S}_2 = \langle s_i \rangle \subseteq W$, $N_i = \pi^{-1}(s_i)$

$\mathrm{Higgs}_{\tilde{X}}(U) := \mathrm{Higgs}'_{\tilde{U}}(U)$.

<u>Rem:</u> Theorem says that $\mathrm{Higgs}_{\tilde{X}}$ is equiv $\mathrm{Higgs}'_{\tilde{X}}$