Let X be a smooth projective variety. For a fixed Bridgeland stability condition, we want to study the moduli functors parameterising (semi)-stable objects living in a derived category and having a fixed numerical Grothendieck class and a fixed phase. In particular, we discuss what is known about the existence of a projective coarse moduli space for these moduli functors for curves and surfaces. In contrast to the case of curves, Coh(X) for a surface X will never be the heart of a Bridgeland stability condition and we need a "Tilting" process to produce a family of Bridgeland stability conditions depending on two parameters. We then turn to the case K3 surfaces and see a result of Toda stating that our moduli functors are Artin stacks of finite type over \mathbb{C}. Some results of Abramovich and Polishchuk are the main ingredients of the proof.

Last time Cesare gave the main ingredients of stability conditions namely the slicing $\mathcal{P}$ and the central charge Z. He defined a topology on the manifold stability conditions through a generalized metric.

He also explained how the $GL^+(2,\mathbb{R})$ action on central charges lifts along the forget map keeping track only of the central charge and this action is an action of the universal cover

$$(T,f)\cdot(\mathcal{P},Z) = (\mathcal{P}(f(\phi)), T^{-1}Z)$$

$$T.Z = T^{-1}Z$$

$$\begin{tikzcd} \widetilde{GL^+(2,\mathbb{R})} \arrow[loop right] & \operatorname{Stab} X \arrow[d] & \ni \sigma = (\mathcal{P},Z) \arrow[d, mapsto, "\mathcal{Z}"] \\ GL^+(2,\mathbb{R}) \arrow[loop right] & \operatorname{Hom}(\Lambda,\mathbb{C}) & \ni Z \end{tikzcd}$$

Th (Bridgeland '07): $\mathcal{Z}: \operatorname{Stab}(X) \longrightarrow \operatorname{Hom}(\Lambda,\mathbb{C})$

is a local homeo. In part. $\operatorname{Stab}(X)$ is a $\mathbb{C}$-manif of dim $\operatorname{rk}\Lambda$

He also gave some computations of $\operatorname{Stab} X$:

E.g. : $\mathrm{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$ [Okada]

- $\mathrm{Stab}(C) \cong \widetilde{GL}^+(2,\mathbb{R})$ for a proj curve C of genus $g \geq 1$.

$\left(\mathrm{Stab}\, C = \sigma_0 \cdot \widetilde{GL}^+(2,\mathbb{R}) \right.$, $\left. \sigma_0 = (\mathrm{Coh}(C), \underbrace{-d+ir}_{= Z_0}) \right)$

- $\mathbb{C}(x)$ & line bundles σ-stable for $\sigma \in \mathrm{Stab}(C)$.

Now using these stability conditions we define today some moduli functors of semistable objects and we study them.

Notation: X : sm. proj var. / $\mathbb{C}$.

$D^b X := D^b(\mathrm{Coh}(X))$

$NS(X) :=$ Neron-Severi group of X

$K_0(X) :=$ Grothendieck group of X.

$K_{num}(X) =$ numerical groth. group $= \dfrac{K_0(X)}{\equiv}$

$E \equiv E' \iff \chi(E,F) = \chi(E',F)$

$\forall F \in K_0(X)$

Part II: parametrize Bridgeland ss-objects:

Let X be a smooth proj var / $\mathbb{C}$,
we have the notion of stability conditions on $D^b(Coh(X))$ in the sense of Bridgeland
In this talk we want to see that the mod stack of ss objects in $D^b(Coh(X))$

with fixed numerical class and phase is represented by an Artin stack

which is oft / $\mathbb{C}$.

Now, we want to parametrize Bridgeland ss objects which are in the derived category so we want to consider families of complexes.

Def: Let $S \in Sch$ l.o.f.t. / $\mathbb{C}$
A complex $\mathcal{F} \in D(QCoh(S \times X))$ is called **S-perfect** if locally over S it is quasi-iso to a bounded complex

$$0 \to \mathcal{F}^m \to \dots \to \mathcal{F}^0 \to 0$$ s.t. $\mathcal{F}^i$ is flat and of p. over S, $\forall i$

Notation: $D_{S\text{-}perf}(S \times X) :=$ The triangulated subcat of S-perfect complexes

Consider the following 2-functor

$$\mathcal{M} : (Sch/\mathbb{C})^{op} \longrightarrow Groupoids$$

$$S \mapsto \{\mathcal{F} \in D_{S\text{-}perf}(S \times X) \mid Ext^{<0}(\mathcal{F}_s, \mathcal{F}_s) = 0 \ \forall s \in S(\mathbb{C})\}$$

$\mathcal{F}_s := \mathcal{F}|_{\{s\} \times X}$

Th: (Lieblich): $\mathcal{M}$ is an alg. stack (Artin stack), l.o.f.t. over $\mathbb{C}$
(i.e. you can find an atlas l.o.f.t)

$\mathcal{M}$ is called "mother of all moduli spaces of sheaves" ☺

So whenever you want study mod. sp. of sheaves, you take this one and then you impose whatever condition you want on your object, then you hope to have a coarse mod. sp.

Rem: [01:10:00]

Let $\mathfrak{M}_{spl} \subset \mathfrak{M}$ be the (open) substack of simple objects (so only complexes w/ scalar endos)

Then $\mathfrak{M}_{spl}$ is again an Artin stack, also l.o.f.t. over $\mathbb{C}$

Th: (Inaba) $M_{spl} : (Sch/\mathbb{C})^{op} \longrightarrow Groupoids \longrightarrow Set$ is representable by an alg. sp. locally of finite type $/\mathbb{C}$

Forget + sheafifify by quotienting by eq rel. given by tensoring w/ line bdles coming from the base S.

After we've seen the notion of family of complexes we can now define the mod functor for Bridgeland ss objects [01:15:00]

Bridgeland ss objects:

$$\sigma := (\mathcal{P}, Z) \qquad \text{s.t.} \qquad Z: K_0(X) \longrightarrow \mathbb{C} \text{ additive hom}$$

central change

slicing

$$\begin{tikzcd} K_0(X) \arrow[r, "Z"] \arrow[dr, "v"'] & \mathbb{C} \\ & \Lambda \arrow[u, "\tilde{Z}"'] \end{tikzcd}$$

s.t.

(1) $\forall\, E \in \mathcal{P}(\phi)$, $Z(E) \in \mathbb{R}_{>0} e^{i\pi\phi}$, $\phi \in (0,1]$
$\;\;E \neq 0$

(2) Support property: $\exists\, Q \in Sym^2 \Lambda^*_{\mathbb{R}}$ s.t. $\forall E \in \mathcal{P}$: $Q(v(E), v(E)) \geq 0$ and Q is neg. def on $\operatorname{Ker} \tilde{Z}$

quad. form on the lattice

– Given $\sigma = (\mathcal{P}, Z)$, $0 \neq E \in \mathcal{P}(\phi)$ is said σ-ss (of phase ϕ)

if moreover E is simple in $\mathcal{P}(\phi)$ then it's said σ-stable

check discussion [14:55], question Mihai

abelian
Noeth + Artinians
$\Downarrow$
$\exists$ JH filt. in simple objects with same phase

Prop: $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(X) \iff (\mathcal{A} := \mathcal{P}((0,1]), Z)$

a numerical class

Grothendieck numerical group

$$\text{Fix } v \in K_{num}(X) = K_0(X) / \{E \in K_0(X) \mid \chi(E,F) = 0 \ \forall F \in K_0(X)\}$$

Fix $\phi \in \mathbb{R}$. a phase

$$\chi : D^b(X) \times D^b(X) \longrightarrow \mathbb{Z}$$
$$(E,F) \longmapsto \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathrm{Hom}(E, F[i])$$

$$\mathcal{M}^s_\sigma(v,\phi) \subset \mathcal{M}^{ss}_\sigma(v,\phi) \subset \mathcal{M}$$

the "mother"

$\mathcal{M}^s_\sigma(v,\phi)$ = substack of Bridgeland σ-stable objects

$\mathcal{M}^{ss}_\sigma(v,\phi)$ = substack of Bridgeland σ-ss objects of type v and phase ϕ (i.e. $E \in \mathcal{P}(\phi)$)

2 natural questions to ask

① Are $\mathcal{M}^s_\sigma(v,\phi) \subset \mathcal{M}^{ss}_\sigma(v,\phi) \subset \mathcal{M}$ open substacks? of finite type over $\mathbb{C}$?

I will focus on the 1st question

② Does $\mathcal{M}^{ss}_\sigma(v,\phi)$ admit a coarse mod space? If yes, is it a projective scheme?

will be addressed in a next talk

[15:23]

before that let me tell you what's known in this generality.

what is known:

① and ② are affirmative for curves ($g \geq 1$)
In this case, the mod spaces are exactly mod. spaces of ss coh. sheaves as we know them in the classical case.

(2) - complete Affirmative Answer for :

$\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $Bl_p \mathbb{P}^2$ it's known how to construct

coarse mod sp. w.r.t stability conditions for these surfaces.

- partial answers for K3 , abelian , Enriquez

and del Pezzo surfaces.

- projectivity for $\mathbb{P}^2$ and two del Pezzo cases were

proved using mod sp. of quiver representations and GIT.

- unfortunately we just have ad-hoc answers like this

because Bridgeland stability is not naturally

a priori associated to a GIT problem.

- once a coarse mod sp. exists, separatedness and

properness follow from a general result by Abramovich

& polishchuk.

(1) We focus on K3 surfaces.

Let me discuss stability conditions on K3 surfaces.

Stability conditions on K3 surfaces (in honor of Kummer, Kähler, Kodaira, K2 mountain in Kashmir)

Def: (K3 surface) X is connected cpct $\mathbb{C}$-surface s.t.

1) $H^1(X, \mathcal{O}_X) = 0$ (in particular, it's simply connected)

2) $\underbrace{\omega_X = \mathcal{O}_X}_{\text{canonical bundle trivial}}$ (for e.g. it's enough to have a holo. symplectic struct on X for this to happen $\exists$ a closed holom. 2-form on X which is non-deg at every pt.)

E.g.: • quartic surface in $\mathbb{P}^3$, $(x^4+y^4+z^4+w^4=0)$

skip • cyclic cover of $\mathbb{P}^2$ branched over a curve of degree 6.

• For surfaces:

. $\mathrm{coh}(X)$ it's not a heart of a Bridg. stab. condition

and $\mathrm{coh}(X)$ is a heart of a bounded t-struct $\leftarrow$.
$\omega, \beta \in NS(X)_{\mathbb{R}}$

$\overline{Z}_{\omega,\beta} : K_0(X) \to \mathbb{C}$

$E \mapsto -\omega^{n-1} \cdot ch_1^{\beta}(E) + i\omega^n \cdot ch_0^{\beta}(E)$

is not a Bridgeland stab. func on $\mathrm{coh}(X)$.

but $\mathrm{coh}(X)$ will never be the heart of a Bridg. stab. cond.

$\overline{Z}_{\omega,\beta}(T) = 0$ for a torsion sheaf T supported in codim ≥ 2

Idea: use tilting to produce a family of stab. cond. depending on ω, β

Def: let $\mathcal{A}$ ab. cat., $\mathcal{F}$, $\mathcal{T}$ are full subcats of $\mathcal{A}$ s.t.

The pair $(\mathcal{F}, \mathcal{T})$ is a torsion pair on $\mathcal{A}$ if:

1) $\forall F \in \mathcal{F}$, $\forall T \in \mathcal{T}$, $\mathrm{Hom}(T, F) = 0$

2) $\forall E \in \mathcal{A}$, $\exists F \in \mathcal{F}$, $\exists T \in \mathcal{T}$ s.t.

$0 \to T \to E \to F \to 0$ a SES. (unique by 1))

Baby e.g.: $\mathcal{A} = \mathrm{Coh}(X)$, X sm. proj. var.

$\mathcal{T} = \{$ torsion sheaves on $X\}$

$\mathcal{F} = \{$ torsion-free sheaves on $X\}$

Operation tilt: given by lemma:

Lemma: (Happel - Reiten - Smalø)

X sm. proj. var.

$\mathcal{A} \subset D^b(X)$ a heart of a bounded t-struct.

$(\mathcal{F}, \mathcal{T})$ torsion pair on $\mathcal{A}$.

we define a subcat

$$\mathcal{A}^\# := \left\{ E \in D^b(X) : \begin{array}{l} H^i_{\mathcal{A}}(E) = 0, \ \forall i \neq 0, -1 \\ H^0_{\mathcal{A}}(E) \in \mathcal{T} \\ H^{-1}_{\mathcal{A}}(E) \in \mathcal{F} \end{array} \right\}$$

H^i: coh. objects w.r.t. the t-struct. $\mathcal{A}$..

$\mathcal{A}^\#$ is a heart of a bounded t-structures on $D^b(X)$.

So now we get a new heart. Alternatively, we can define $\mathcal{A}^\#$ as the smallest extensions-closed full subcat of $D^b(X)$ containing $\mathcal{F}[1], \mathcal{T}$:

$\boxed{\mathcal{A}^\# = \langle \mathcal{F}[1], \mathcal{T} \rangle}$ (tilted heart)

Let's now define this family of stability conditions using the divisor classes
But before we need to do a small digression and make some definition

Let X a K3 surface. (let's focus on the K3 surfaces)

Mukai Vectors: Let $NS^*(X)$ be the alg Mukai lattice

$$NS^*(X) = H^0(X,\mathbb{Z}) \oplus NS(X) \oplus H^4(X,\mathbb{Z}) \cong \mathbb{Z} \oplus NS(X) \oplus \mathbb{Z}$$

Mukai V pairing (bilinear): $\big((r, c, s), (r', c', s')\big)_{Muk} = cc' - rs' - r's$

$$v : K_{num}(X) \xrightarrow{\cong} NS^*(X)$$

$$E \longrightarrow v(E) = ch(E).\sqrt{td_X}$$

$$= (r(E), c_1(E), ch_2(E) + r(E))$$

Mukai vector map, a modified version of the chern character

sending an obj to it's Mukai vector gives an iso sending

$$-\chi(E, E') \longleftrightarrow (v(E), v(E'))_{Muk}$$

Let $\omega, \beta \in NS(X)_{\mathbb{Q}}$, ω ample. we will construct a family of stability conditions dep on these two divisor classes

$$Z_{(\beta,\omega)} = (e^{\beta + i\omega}, v(E))_{Muk} : K_{num}(X) \to \mathbb{C}$$

stab fun

Mukai pairing in $\mathbb{C}$. $NS^*(X)_{\mathbb{C}}$

Mukai vector

if $v(E) = (r, c, s)$, $r \neq 0$ then $Z_{(\beta,\omega)}(E) = \frac{1}{2r}\big((c^2 - 2rs) + r^2\omega^2 - (c - r\beta)^2\big) + i(c - r\beta).\omega$

if $r = 0$ then $Z_{\beta,\omega}(E) = (-\Delta + c\beta) + i(c\omega)$.

<u>Recall</u>: twisted slope-stab. on $\mathrm{Coh}(X)$.

for $v(E) = (r, c, \Delta)$ with $r > 0$, define:

$$\mu_\omega(E) := \frac{c \cdot \omega}{r}$$

(where dividing by 0 is just $+\infty$)

A sheaf $E \in \mathrm{coh}(X)$ is μ_ω-(semi)-stable if $\forall F \subsetneq E$, we have

$$\mu_\omega(F) \underset{(\leq)}{<} \mu_\omega(F)$$

$\forall E \in \mathrm{Coh}(X)$, we have the Notion of HN-filtrations and so on

$\mathcal{T}_{\omega,\beta} \subset \mathrm{Coh}(X)$: sheaves whose Torsion-free part have μ_ω-ss HN-factors s.t. $\mu_{B,\omega} > \beta . \omega$.

$\mathcal{F}_{\omega,\beta} \subset \mathrm{Coh}(X)$: Torsion free sheaves whose μ_ω-ss HN-factors s.t. $\mu_\omega \leq \beta . \omega$

Skip

$$\mathcal{A}_{\beta,\omega} = \{ E \in D^b(X) \mid H^{-1}(E) \in \mathcal{F}_{(\omega,\beta)},\ H^0(E) \in \mathcal{T}_{(\beta,\omega)} \}$$

or Recall the Tilting: $\mathcal{A}_{\beta,\omega} = \langle \underbrace{\mathcal{F}_{\omega,\beta}[1]}_{\text{torsion-free part}}, \underbrace{\mathcal{T}_{\omega,\beta}}_{\text{torsion part}} \rangle$

<u>Rem</u>: Note that for diff choices of β, ω, we can still have the same cat $\mathcal{A}_{\beta,\omega}$,

e.g. $\mathcal{A}_{\beta,k\omega} = \mathcal{A}_{\beta,\omega}$ for $k \in \mathbb{Q}_{\geq 1}$.

skip

$$0 \to \mathcal{F}[1] \to E \to \mathcal{T} \to 0$$

$$\mathrm{Ext}[E,E]$$

(whenever you have sth from torsion to torsion free, it's 0, this would prove $\mathrm{Ext}(\mathcal{E}, E[1]) = 0$)

Th (Bridgeland): $(\mathcal{A}_{\beta,\omega}, \underbrace{Z_{\beta,\omega}}_{\text{central change}}) =: \sigma_{\beta,\omega}$ is a Bridgeland stability condition iff $\forall E$ a spherical sheaf on X, $Z_{\beta,\omega}(E) \notin \mathbb{R}_{\leq 0}$.

In part. this holds for $\omega^2 > 2$ self intersection (prop. 4.4. (Toda))

(E spherical $\iff \mathrm{End}(E) = \mathbb{C}$ and $\mathrm{Ext}^1(E,E) = 0$)

Let $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ be the connected component containing $\sigma_{\beta,\omega}$ all these stability conditions

(th. 4.12 Toda)

(Toda) Th: Let X be a K3 surface, let $\sigma \in \mathrm{Stab}^*(X)$, $v \in K_{num}(X)$, $\phi \in \mathbb{R}$. Then $\mathcal{M}^{ss}_{\sigma}(v,\phi)$ is an alg. stack of finite type$/\mathbb{C}$

Toda makes several assumptions in order to prove the th, then he proves that these assumptions are true for the case of moduli functor of Bridg. ss objects for K3 surfaces

__Assumptions__: $\mathcal{M}^{ss}_{\sigma}(v,\phi)$ is bounded and open in $\mathcal{M}$.

__Proof idea__: Then he observes that if $M \to \mathcal{M}$ is an atlas of $\mathcal{M}$

openness $\Rightarrow \exists\, M^{\circ} \underset{\text{open}}{\subset} M$ & a smooth $M^{\circ} \twoheadrightarrow \mathcal{M}^{ss}_{\sigma}(v,\phi)$

boundedness $\Rightarrow \exists\, M' \twoheadrightarrow M^{\circ}$, M' a f.t. $\mathbb{C}$-scheme.

$\rightsquigarrow M^{\circ}$ o.f.t. and gives an atlas of $\mathcal{M}^{ss}_{\sigma}(v,\phi)$ $\blacksquare$

For time reasons, let's assume boundedness and focus on openness.

skip

Let $S \in Sch/\mathbb{C}$, $\mathcal{F} \in \mathcal{M}(S)$

is $\{ s \in S \mid \mathcal{F}_s \in D^b(X)$ of numerical type v and $\mathcal{F}_s \in \mathcal{P}(\phi) \}$ open?

according to Toda. An affirmative answer gives a sufficient condition for openness

we need the following th:

Th: (Abramovich, Polishchuk) noted (A-P) below

Let $\mathcal{A} \subset D^b(X)$ a heart of a bounded t-structure s.t. $\mathcal{A}$ is noetherian and let S be a smooth proj. variety with $\mathcal{L} \in \mathrm{Pic}(S)$ ample; then the subcat:

$$\mathcal{A}_S = \{ F \in \overbrace{D^b(X \times S)}^{\text{bounded complexes}} \mid \overbrace{R}^{\text{derived push-forward}}_{p*}(F \otimes q^*\mathcal{L}^m) \in \mathcal{A}, m \gg 0 \} \text{ is a}$$

$$X \xleftarrow{p} X \times S \xrightarrow{q} S$$

heart of a bounded t-structure, indep of a choice of $\mathcal{L}$

Moreover, $\mathcal{A}_S$ is an abelian noeth. category.

(a heart is always abelian)

Lemma 3.6 (Toda): openness of $\mathcal{M}_\sigma(v,\phi)$ in $\mathcal{M}$ (the mother of all ...)

reduces to the following (generic flatness problem) (prop 3.10)

$\forall$ S smooth proj. scheme, $\forall \mathcal{E} \in \mathcal{A}_S$ (family), $\exists ? U$ open $\subset S$ s.t.

$\mathcal{E}_s \in \mathcal{A} \quad \forall s \in U.$

It turns out there is a partial result for the generic flatness pb:

(A-P): $\forall \mathcal{E} \in \mathcal{A}_S$, $\exists$ $U \subset S$ dense s.t. $\mathcal{E}_s \in \mathcal{A}$, $\forall s \in U$

(prop 3.12)

$\mathcal{A} = \mathcal{A}_{p,\omega}$ (tilted heart)

$\mathcal{A} = \mathcal{P}((0,1])$ if you need ϕ not in $[0,1)$, just change the heart.

Th: (Toda) (lemma 4.7)

generic flatness pb is true for $\mathcal{A} = \mathcal{A}_{\omega,\beta}$ (tilted heart).

more explicitly: $\exists \{ s \in S \mid \mathcal{E}_s \in \mathcal{A}_{\beta,\omega} \}$ open for the Zariski top.

Proof: Let S be smooth proj $\mathbb{C}$-var, $\mathcal{L} \in Pic(S)$ ample

let $\mathcal{E} \in \mathcal{A}_S$

$Rp_*(\xi \otimes \mathcal{L}^m) \in \mathcal{A}_{\beta,\omega}$ for $m \gg 0$ by def of $\mathcal{A}_S$ in A-P theorem

In particular $\rightsquigarrow$ Rp_* concentrated in degree $-1, 0$. (Tilting)

Now consider the spectral sequence:

$$E_2^{i,j} = R^i_{p_*}\left(H^j(\mathcal{E}) \otimes \mathcal{L}^m\right) \Rightarrow R^{i+j}_{p_*}(\mathcal{E} \otimes \mathcal{L}^m) \text{ degenerates for } m \gg 0$$

$$\Rightarrow H^j(\mathcal{E}) = 0, \ \forall\, j \neq \{-1, 0\}$$

$$\Rightarrow \exists\, U \overset{\text{open}}{\subset} S$$

- $0 = F^k \subseteq \dots \subseteq F^0 = H^{-1}(\mathcal{E})|_U$
- $0 = T^l \subseteq \dots \subseteq T^0 = H^0(\mathcal{E})|_U$

$\exists$ Relative HN filtration of coh. sheaves (For. e.g. from the book of Manfred-Lehn, Geom. of mod spaces)

s.t.

- $F^i/F^{i+1}, T^i/T^{i+1}$ are U-flat, and if you restrict to each fiber:

- $$\left.\begin{array}{l} F^k_s \subseteq \dots \subseteq F^0_s = H^{-1}(\mathcal{E})_s \\ T^l_s \subseteq \dots \subseteq T^0_s = H^0(\mathcal{E})_s \end{array}\right\} \text{are}$$ absolute HN filtrations (for slope μ_ω),

$$\mathcal{E}_s \in \mathcal{A}_{\omega,\beta} \iff \mu_\omega(F^k_s) \leq \omega\cdot\beta, \ \mu_\omega(T^0_s/T^1_s) > \beta\omega \text{ or } H^0(\mathcal{E})_s \text{ is torsion}$$ (☆)

- $(A\text{-}P) \Rightarrow \{ s \in S \mid \mathcal{E}_s \in \mathcal{A}_{\omega,\beta} \}$ is dense

$\rightsquigarrow$ (when you have a flat family the slope will not change)

$\forall i, \forall s, s' \in U$, F^i_s, T^i_s are numerically equivalent to $F^i_{s'}$ and $T^i_{s'}$ respectively. Hence $\forall s \in U$, (☆) is true $\blacksquare$

$\forall s \in U, \quad \mathcal{E}_s \in \mathcal{A}_{\beta,w}$

Locally around $s \in U$. $\mathcal{E}_s \in \mathcal{A}_{\beta,w}$

For the next talk, we will study the wall and chamber structure of $\operatorname{Stab}(X)$. We'll see that if we fix a numerical class v, we can check that $\exists$ loc. finite wall and chamber struct s.t. the set of ss obj. of class v is constant within each chamber, so the ss obj vary nicely within $\operatorname{Stab}(X)$.