Talk 3:

06/12/2022

0) Setup:

$k = \mathbb{F}_q$ be a field, $\bar{k}$: its alg. closure

Let X be a smooth projective, geometrically connected curve $/k$, $\bar{X} = X \times_k \bar{k}$.

Fix D be an effective divisor

Let $\mathbb{G}$ be a connected, reductive over k, W it's weyl group, s.t. $\mathrm{char}(k) \nmid |W|$.

(the weyl grp is the grp of automorphisms of $\mathbb{G}$ generated by reflections of the root system of $\mathbb{G}$ w.r.t. it's maximal torus.)

Let G a grp scheme on X s.t. $G \cong \mathbb{G} \times X$ étale-locally on X.

the underlying scheme is Étale-loc a number of copies of X indexed by elements of $\mathbb{G}$: For $\coprod_i U_i \twoheadrightarrow X$, $G|_{U_i} = \coprod_{g \in \mathbb{G}} X$

and we can define the multiplication map:

$$G|_{U_i} \underset{X}{\times} G|_{U_i} := \coprod_{(g,h) \in \mathbb{G}^2} X \xrightarrow{m} \mathbb{G}|_{U_i}$$

$$(g,h)\text{-copy of } X \overset{id}{\longmapsto} (gh)\text{-copy of } X$$

then define the inverse and unity maps in the obvious way.

Rem: G constructed this way is flat algebraic étale locally X

i.e. étale locally, G is affine + o.f.p. over X

Def: A splitting (épinglage) of $\mathbb{G}$ over k is the data:

- $\mathbb{T} \subset \mathbb{G}$ a maximal torus over k.
- $\mathbb{B} \subset \mathbb{G}$ a Borel subgrp over k s.t. $\mathbb{T} \subset \mathbb{B}$.

$\Big\}$ "quasi-splitting"

- $\forall \alpha \in \Delta \subset \Phi$ (simple root of the root system Φ of G w.r.t. $\mathbb{T}$), a vector $x_\alpha \neq 0$ from the eigenspace $\mathfrak{g}_\alpha$ of $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$ where $\mathbb{T}$ acts by the character α.

- From now, fix a splitting of $\mathbb{G}$ over k.

(every connected reductive grp over a finite field is quasi-split)

Notation

- $\mathbb{G}$ split $\Longrightarrow$ x_α are defined over k.
- $x_+ := \sum_{\alpha \in \Delta} x_\alpha$.
- $\forall \alpha \in \Delta$, choose! $x_{-\alpha} \neq 0$ in the eigenspace of $\mathbb{T}$ in $\mathfrak{g}$ of character $-\alpha$ s.t. $[x_\alpha, x_{-\alpha}] = \alpha^\vee$.

  Denote $x_- := \sum_{\alpha \in \Delta} x_{-\alpha}$.
- $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$, $\mathfrak{t} := \mathrm{Lie}(\mathbb{T})$, $\mathfrak{g} := \mathrm{Lie}(G)$
- $\mathrm{Car}_{\mathbb{G}} := \mathrm{Spec}(k[\mathfrak{t}]^W)$ ($W \curvearrowright \mathfrak{t}$ is the restriction of the adjoint $\mathbb{G} \curvearrowright \mathfrak{g}$)

I) Chevalley morphism and the section of Kostant:

our starting point will be a statement of Kostant in char 0

Th: Kostant (char 0), Veldkamp (char p, p big enough w.r.t. root system)

(More precisely) If $\mathrm{char}(k) \nmid |W|$, then

1) $\mathfrak{g} \to \mathfrak{t} \rightsquigarrow k[\mathfrak{g}]^G \xrightarrow{\sim} k[\mathfrak{t}]^W = \underbrace{k[\mathrm{car}_G]}_{\text{coordinate alg. of } \mathrm{car}_G}$.

Moreover, $k[\mathfrak{t}]^W$ is an algebra of homogeneous polynomials in variables $u_1, \dots, u_r$, of degrees $m_1+1, \dots, m_r+1$ (There is no really good way to choose the homog. polys but the degrees m_i are indep. of any choices and are usually called "exponents of G")

$\rightsquigarrow$ a $\mathbb{G}_m$-equiv. map: $\chi : \mathfrak{g} \to \mathrm{car}_G$ "Chevalley morph"

with $\mathbb{G}_m \circlearrowright k[\mathfrak{t}]^W$ via: $t.(u_1, \dots, u_r) = (t^{m_1+1} u_1, \dots, t^{m_r+1} u_r)$

($\mathbb{G}_m \circlearrowright \mathfrak{g}$ by homothety)

2) Let $\mathfrak{g}^{reg} := \{ x \in \mathfrak{g} : \dim \underbrace{C_G}_{\text{centraliser}}(x) = r \} \overset{\text{open}}{\subset} \mathfrak{g}$ (for the top. induced by any norm on the f.d. v.s. $\mathfrak{g}$)

Then $\underbrace{\mathfrak{g}^{reg} \to \mathrm{car}_G}_{\text{restriction}}$ is smooth whose fibers are G-orbits. (for adj. action)

3) Let $\mathfrak{g}^{x_+} := \mathrm{Lie}(C_G(x_+))$. Then the affine subspace

$x_- + \mathfrak{g}^{x_+} \subset \mathfrak{g}^{reg}$ and $\chi|_{x_- + \mathfrak{g}^{x_+}} : x_- + \mathfrak{g}^{x_+} \xrightarrow{\cong} \mathrm{car}_G$

call its inverse the Kostant section: $\varepsilon : \mathrm{car}_G \to \mathfrak{g}^{reg}$

II) Stacky Hitchin map:

using the stacks language to construct the Hitchin map may appear cumbersome but it will reveal itself very useful, at least if one wishes to treat reductive group schemes uniformly and not case by case

classical GL_m case:

Let (E, φ) a Hitchin pair, E : a rank-m V.B. on X

φ : a twisted endo $\varphi : E \to E \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) =: E(D)$

$\overset{\text{taking}}{\underset{\text{exterior product}}{\rightsquigarrow}}$ $\Lambda^i \varphi : \Lambda^i E \to \Lambda^i E \otimes_{\mathcal{O}_X} \mathcal{O}_X(iD)$

$\overset{\text{taking}}{\underset{\text{trace}}{\rightsquigarrow}}$ $\mathrm{tr}(\Lambda^i \varphi) \in H^0(X, \mathcal{O}_X(iD))$

$\overset{\text{this}}{\underset{\text{defines}}{\rightsquigarrow}}$ Hitchin map: $f : \mathcal{M} \to \mathbb{A} := \bigoplus_{i=1}^{m} H^0(X, \mathcal{O}_X(iD))$

we will now define the Hitchin map more generally for G any reductive group over k.

Def: A **Hitchin pair** (E, φ) w.r.t. X, G and $\mathcal{O}_X(D)$ consists of a

- G-torsor $E \to X$
- $\varphi \in H^0(X, \mathrm{ad}(E) \otimes \mathcal{O}_X(D))$, $\mathrm{ad}\, E := E \times_{\mathrm{ad}} \mathfrak{g} \longrightarrow X$

V.B. obtained by pushing out E by the adj. rep. of G

- an algebraic stack of Higgs pairs defined via its groupoid of sections assigning to each test scheme S the Cat. of Hitchin pairs parametrised by S.

$$\mathcal{M} : \left(\mathrm{Sch}/_k\right)^{op} \ni S \longmapsto \left\{ (E, \varphi) \;\middle|\; \begin{array}{l} E \to X \times S \text{ a } G\text{-torsor} \\ \varphi \in H^0(X \times S, \mathrm{ad}(E) \otimes pr_X^* \mathcal{O}_X(D)) \end{array} \right\}$$

(this stack is algebraic simply because the moduli of G-bundles is an algebraic stack) let $(E,\varphi) \in \mathcal{M}(S)$ an S-valued point, $S \in Sch/_k$

$E \rightsquigarrow$ continuous $h_E : X \times S \longrightarrow BG$ (BG the classifying space)

similarly, $\mathcal{O}_X(D) \rightsquigarrow h_D : X \longrightarrow B\mathbb{G}_m$

(every line bundle $L \to X$ give a $\mathbb{G}_m$-torsor
multiplicative group
$L \setminus Z(X)$: the complement of 0 section)

$\rightsquigarrow h_E \times h_D : X \times S \longrightarrow BG \times B\mathbb{G}_m$

Technical Lemma: let G algebraic grp, S a base scheme,
(Kitchen exercise)

$V \in Sch/_S$, $G \circlearrowright V$, let $T \in Sch/_S$

Given a G-torsor $E \to T$, Then

$$\left(\pi : E \to V \quad G\text{-equiv}\right) \iff \left(\varphi \in H^0(T, E \times^G V)\right)$$

In particular, $[V/G]$ classifies G-torsors $E \to T$ endowed with a global section $\varphi \in H^0(T, E \times^G V)$

The proof is not difficult so we will directly apply it to our case, but the important consequence here

$V := \mathfrak{g} \rightsquigarrow$

$$\begin{tikzcd}
\mathrm{ad}(E) \otimes \mathcal{O}_X(D) \arrow[d] & & {[\mathfrak{g} / G \times \mathbb{G}_m]} \arrow[d] \\
X \times S \arrow[u, bend right, "\varphi"'] \arrow[urr, dashed, "h_{E,\varphi}"] \arrow[rr, "h_E \times h_D"'] & & BG \times B\mathbb{G}_m
\end{tikzcd}$$

$\mathfrak{g}$ { $G \circlearrowright \mathfrak{g}$ adj. act. and $\mathbb{G}_m \circlearrowright \mathfrak{g}$ by homothety

$\varphi \in H^0(X\times S, \mathrm{ad}(E)\otimes \mathcal{O}_X(D)) \overset{\text{technical}}{\underset{\text{Lemma}}{\Longleftrightarrow}} h_{E,\varphi}$ lifting $h_E \times h_D$

The data of $h_{E,\varphi}$ determines (E,φ) and vice versa.

Let $x \in X$ a geometric point.

$G \rightsquigarrow$ an $\mathrm{Aut}(\mathbb{G})$-torsor $\mathcal{T}_G \longrightarrow X$ s.t.

isomorphism class is $[\mathcal{T}_G] \in H^1(\pi_1(X,x), \mathrm{Aut}(\mathbb{G})(\bar{k}))$

$$\rightsquigarrow G = \mathbb{G} \overset{\mathrm{Aut}(\mathbb{G})}{\times} \mathcal{T}_G, \quad \mathfrak{g} = \mathfrak{g} \overset{\mathrm{Aut}(\mathbb{G})}{\times} \mathcal{T}_G$$

and $[\mathfrak{g}/G\times\mathbb{G}_m] = [\mathfrak{g}/\mathbb{G}\times\mathbb{G}_m] \overset{\mathrm{Aut}(\mathbb{G})}{\times} \mathcal{T}_G$ (strict action $\mathrm{Aut}(\mathbb{G}) \circlearrowright [\mathfrak{g}/_{G\times\mathbb{G}_m}]$)

Also: $\mathbb{G}/_{Z(\mathbb{G})}$

$$1 \to \underbrace{\mathrm{Int}(\mathbb{G})}_{\mathbb{G}/Z(\mathbb{G})} \to \mathrm{Aut}(\mathbb{G}) \to \overbrace{\mathrm{Out}(\mathbb{G})}^{\text{a discrete group}} \to 1$$

and $\left.\begin{array}{l}\mathrm{Out}(\mathbb{G}) \circlearrowright k[\mathfrak{t}] \\ \mathrm{Out}(\mathbb{G}) \circlearrowright W\end{array}\right\} \Longrightarrow \mathrm{Out}(\mathbb{G}) \circlearrowright \mathrm{Car}_{\mathbb{G}}$.

$\rightsquigarrow \mathrm{Aut}(\mathbb{G}) \circlearrowright \mathrm{Car}_{\mathbb{G}}$ though $\mathrm{Out}(\mathbb{G})$.

$\rightsquigarrow \mathrm{Car} := \mathrm{Car}_{\mathbb{G}} \overset{\mathrm{Aut}(\mathbb{G})}{\times} \mathcal{T}_G$. (we can twist $\mathrm{Car}_{\mathbb{G}}$ by $\mathcal{T}_G$)

on the other hand $\mathrm{Aut}(\mathbb{G}) \circlearrowright \mathbb{G} \rightsquigarrow \mathrm{Aut}(\mathbb{G}) \circlearrowright \mathfrak{g}$ (tautological action)

$\rightsquigarrow \chi : \mathfrak{g} \to \mathrm{Car}_{\mathbb{G}}$ is $\mathrm{Aut}(\mathbb{G})$-equiv. w.r.t. to above actions.

But from Kostant–Veldkamp, χ is $\mathbb{G}_m$-equiv.

$\rightsquigarrow$ χ is $\mathrm{Aut}(G) \times \mathbb{G}_m$-equiv.

induces $\rightsquigarrow$ $\mathfrak{g} \longrightarrow \mathfrak{car}$ $\quad G \times \mathbb{G}_m$-equiv (after twisting by $\widetilde{T}_G$)

$\rightsquigarrow$ $\mathfrak{g}/G \longrightarrow \mathfrak{car}/\mathbb{G}_m$ is $\mathbb{G}_m$-inv.

$\rightsquigarrow$ $[\chi] : [\mathfrak{g}/(G \times \mathbb{G}_m)] \longrightarrow [\mathfrak{car}/\mathbb{G}_m]$ a quotient stack morph.

$$\begin{tikzcd}
X \times S \arrow[r, "h_{E,\varphi}"] \arrow[rr, bend left, "h_a"] \arrow[dr, "h_E \times h_D"'] \arrow[drr, "h_D"] & {[\mathfrak{g}/(G \times \mathbb{G}_m)]} \arrow[r, "{[\chi]}"] \arrow[d] & {[\mathfrak{car}/\mathbb{G}_m]} \arrow[d] \\
 & BG \times B\mathbb{G}_m \arrow[r] & B\mathbb{G}_m
\end{tikzcd}$$

our stacky Hitchin map sends a pair (E, φ) which is the same thing as $h_{E,\varphi}$ to a an arrow ha above h_D.

Let's make a functor out of this.

Representability Lemma: the functor $H : Sch/_k \longrightarrow Cat$
(Lemma . 2.4)

$$S \longmapsto \left\{ \begin{array}{l} \text{arrows } h_a : X \times S \to [\mathfrak{car}/\mathbb{G}_m] \\ \text{above } h_D : X \times S \to B\mathbb{G}_m \end{array} \right\}$$

is representable by an affine space $\mathbb{A}$ called the Hitchin affine space. Thus we get the Hitchin map $h : \mathcal{M} \to \mathbb{A}$.

"Proof:" $\mathcal{O}_X(D) \longleftrightarrow$ a $\mathbb{G}_m$-torsor $L_D \longrightarrow X \times S$

giving h_a is equivalent to giving a section a

$$h_a \xLeftrightarrow{\text{tech. Lemma}} a : X \times S \longrightarrow \mathrm{car} \times^{\mathbb{G}_m} L_D$$

hence our subscript a in h_a

$\mathrm{car} \times^{\mathbb{G}_m} L_D$ is the total space of a V.B. over X

since it's obtained by twisting the affine space $\mathrm{car}_{\mathbb{G}}$.

Hence H is representable by $\mathbb{A} := H^0(X, \mathrm{car} \times^{\mathbb{G}_m} L_D)$ ▨

Now, obviously we want to recover the original Hitchin description of $\mathbb{A}$.

<u>Rem:</u> if $G \cong X \times \mathbb{G}$ (the constant grp),

$$\mathrm{car} \times^{\mathbb{G}_m} L_D = \left| \bigoplus_{i=1}^{r} \mathcal{O}_X((m_i+1)D) \right|$$ total space

its space of global sections is:

$$A(k) := \bigoplus_{i=1}^{r} H^0(X, \mathcal{O}_X((m_i+1)D))$$ (original Hitchin description of this affine space) ▨

The most basic aspect of studying the Hitchin map is to be able to predict when is a fibre $\mathcal{M}_a(k)$ non-empty?

Prop 2.5: Suppose that G is quasi-split.

Given $L_D^{\otimes 1/2}$ a square root of L_D, then there exists a section of h.

Proof idea: construct a section of the Hitchin map modeled on the Kostant section. $\blacksquare$

III) Centralisers

: this is a rather technical section but there is a motivation behind

- Motivation : Just like the moduli of V.B., the alg. stack $\mathcal{M}$ is not o.f.t.
Nevertheless, the authors could describe k-points of $\mathcal{M}_a$ in adelic terms in the same way Weil counted the V.B., so broadly speaking
Adelic point counting for $|\mathcal{M}_a(k)|$ suggests an action of some kind of group on $\mathcal{M}_a$. It turns out that it's an action of a Picard stack $\mathcal{P}_a$
The goal of this section is to prepare the definition of $\mathcal{P}_a$.

- take the group scheme over $\mathfrak{g}$ of centralisers

$$I_G = \{(x,g) \in \mathfrak{g} \times G \mid ad(g)x = x\}$$

- $G \curvearrowright \mathfrak{g}$ (adjoint action) :

$$\begin{aligned} \mathfrak{g} \times G &\longrightarrow \mathfrak{g} \\ (x,g) &\longmapsto ad(g)x := gxg^{-1} \end{aligned}$$

lifts $\rightsquigarrow$ to I_G

$$\begin{aligned} G \times I_G &\longrightarrow I_G \\ (h,(x,g)) &\longmapsto (ad(h)x, ad(h)g) \end{aligned}$$

$\mathbb{G}_m \curvearrowright \mathfrak{g}$ (homothety)

$$\begin{aligned} \mathbb{G}_m \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (t,x) &\longmapsto tx \end{aligned}$$

lifts $\rightsquigarrow$ to I_G

$$\begin{aligned} \mathbb{G}_m \times I_G &\longrightarrow I_G \\ (t,(x,g)) &\longmapsto (tx, g) \end{aligned}$$

$$\mathrm{Aut}(\mathbb{G}) \curvearrowright$$

group $\rightsquigarrow$ scheme $\quad \boxed{[I_{\mathbb{G}}]} := [I_{\mathbb{G}} / \mathbb{G} \times \mathbb{G}_m] \longrightarrow [\mathfrak{g} / \mathbb{G} \times \mathbb{G}_m] \quad \mathrm{Aut}(\mathbb{G}) \circlearrowright$

group $\rightsquigarrow$ scheme $\quad \boxed{[I]} := [I_{\mathbb{G}}] \times^{\mathrm{Aut}\,\mathbb{G}} \tau_{\mathbb{G}} \longrightarrow [\mathfrak{g}/G \times \mathbb{G}_m]$

__Lemma 3.1.__ (tautological)

Let $(E, \varphi) \in \mathcal{M}(S)$ a Hitchin pair $\longleftrightarrow h_{E,\varphi}$

Let $h_{E,\varphi} : X \times S \longrightarrow [\mathfrak{g} / G \times \mathbb{G}_m]$ the corresponding map by the technical lemma.

Then: $\underbrace{I_{E,\varphi}} = h^*_{E,\varphi} [I]$

grp scheme representing the sheaf of automorphisms of (E, φ) over $X \times S$

__Prop 3.2__: $\exists$ a group scheme $J_{\mathbb{G}}$ over $\mathrm{Car}_{\mathbb{G}}$ s.t. $\exists$ canonical iso $\chi^* J_{\mathbb{G}}|_{\mathfrak{g}^{reg}} \xrightarrow{\sim} I_{\mathbb{G}}|_{\mathfrak{g}^{reg}}$ which extends to a morphism of grp schemes $\chi^* J_{\mathbb{G}} \longrightarrow I_{\mathbb{G}}$.

__Proof__: Recall $\chi : \mathfrak{g} \to \mathrm{Car}_{\mathbb{G}}$ the Chevalley morphism.

and $\varepsilon : \mathrm{Car}_{\mathbb{G}} \xrightarrow{\kappa} \mathfrak{g}^{reg} \hookrightarrow \mathfrak{g}$ the Kostant section

$I_G \to \mathfrak{g}$ is smooth above $\mathfrak{g}^{reg}$

$\xrightarrow{\quad}$ $J_G := \varepsilon^* I_G \longrightarrow Car_G$ is smooth of relative dimension r.

car(k) ~~/ |W|~~

consider the map: $\beta : G \times Car_G \longrightarrow \mathfrak{g}^{reg}$

$(g, a) \longmapsto ad(g)\varepsilon(a)$

$\uparrow$ projection

well defined: adjoint action preserves $\mathfrak{g}^{reg}$

β is smooth + surj so faithfully flat.

$I_G|\mathfrak{g}^{reg} = \{(x,h) \in \mathfrak{g} \times G : ad(h).x = x\}$

$$\beta^{-1}(I_G|\mathfrak{g}^{reg}) = \{(g, a ; x, h) : ad(g)\varepsilon(a) = x = ad(h)x = ad(h)(ad(g)\varepsilon(a))\}$$

$$= \{(g, a; h) : ad(g)\varepsilon(a) = ad(hg)\varepsilon(a)\}$$

$$\beta^{-1}(\chi^* J_G|\mathfrak{g}^{reg}) = \{(g, a ; h') : \varepsilon(a) = ad(h')\, \varepsilon(a) \}$$

(easily seen details in help 10)

clearly $\rightsquigarrow$ an iso

$$\beta^* \chi^* J|\mathfrak{g}^{reg} \xrightarrow[u]{\sim} \beta^* I|\mathfrak{g}^{reg}$$

$$(g, a; h') \longmapsto (g, a; h), \quad h = g h' g^{-1}$$

As you certainly gessed, since β is faithfully flat, we will use faithfully flat descent to prove that this iso descends along β.

$\rightsquigarrow$ Enough to show a cocycle equality on

$$\underbrace{(G \times Car_G) \times_{\mathfrak{g}^{reg}} (G \times Car_G)}_{\substack{\shortparallel \\ \mathcal{E}}} = \{(g_1, g_2, a) : ad(g_1)\varepsilon(a) = ad(g_2)\varepsilon(a)\}$$

$$= \{(g_1, g_2) : (g_1^{-1} g_2) \in \overbrace{I_{\varepsilon(a)}}^{\text{centraliser of } \varepsilon(a)}\}$$

the two pullbacks of v to $\mathcal{E}$ differ by the interior automorphism

$$\mathrm{int}(g^{-1} g_2) \in I_{\mathcal{E}(a)}$$

so $\mathrm{int}(g^{-1} g_2) = \mathrm{id}$ ($I_{\mathcal{E}(a)}$ is a commutative group)

by Grothendieck's faithfully flat descent: $\chi^* J_G|_{\mathfrak{g}^{reg}} \xrightarrow{\sim} I_G|_{\mathfrak{g}^{reg}}$.

This proved the 1st part of the proposition.

Now this is only a morphism of sheaves. to see it's really an iso of grp schemes, recall that

$J_G \longrightarrow \mathrm{car}_G$ is smooth. $\leadsto$ $\chi^* J_G \to \mathfrak{g}$ is a smooth grp scheme (in particular normal) ①

$(\mathfrak{g} \setminus \mathfrak{g}^{reg}) \subset \mathfrak{g}$ is closed of codim $3 \geq 2$.

$\Rightarrow (\chi^* J_G \setminus \chi^* J_G|_{\mathfrak{g}^{reg}}) \subset \chi^* J_G$ is closed of codim ≥ 2. ②

By [EGA IV, part 4, 20.4.12] which deals with local study of morphisms of schemes, we have that

① + ② $\Rightarrow$ $\chi^* J_G|_{\mathfrak{g}^{reg}} \longrightarrow I_G|$ extends to a unique morphisme $\chi^* J_G \longrightarrow I_G$ of grp schemes over $\mathfrak{g}$. $\blacksquare$

Let's write this result in the stacky language.

Prop 3.3: $\exists$ group scheme $[J]$ over $[\mathrm{car}/\mathbb{G}_m]$ unique up to unique iso, s.t. its inverse image over car is J.

Moreover, on $[\mathfrak{g}/(G \times \mathbb{G}_m)]$, we have a canonical morphism $[\chi]^*[J] \longrightarrow [I]$ whose restriction to $[\mathfrak{g}^{reg}/G \times \mathbb{G}_m]$ is an iso.

"Proof": idea: reuse prop 3.2, and twist by the $\mathrm{Aut}(G)$-torsor $\mathcal{T}_G$ $\square$

Prop. 3.4: the morphism $[\mathfrak{g}^{reg}/G] \longrightarrow \mathrm{car}_G$ is a J-gerbe.

"Proof": $\mathrm{car}_G = \mathrm{Spec}\, k[\mathfrak{t}]^W \cong \mathrm{Spec}\, k[\mathfrak{g}]^G$

so the G-inv morph $\chi|_{\mathfrak{g}^{reg}} : \mathfrak{g}^{reg} \longrightarrow \mathrm{car}_G$ corresponds to a morph $[\mathfrak{g}^{reg}/G] \longrightarrow \mathrm{car}_G$.

The statement is same as proving that $[\mathfrak{g}^{reg}/G] \xrightarrow{\sim} [\mathrm{car}_G/J]$

Recall β: $G \times \mathrm{car}_G \longrightarrow \mathfrak{g}^{reg}$ is smooth and surj.

$$(g, a) \longmapsto \mathrm{ad}(g)\,\varepsilon(a)$$

So $J = \varepsilon^* \overset{\text{group scheme of centralisers}}{\overbrace{I}} \Longrightarrow (G \times \mathrm{car}_G)/J \xrightarrow{\sim} \mathfrak{g}^{reg}$

so dividing by the action of G we get

$$[\mathrm{car}_G/J] \xrightarrow{\sim} [\mathfrak{g}^{reg}/G]$$

$\square$

Now we're ready to discuss the Picard stack over the fibers of Hitchin map.

IV $\mathcal{P}_a \curvearrowright \mathcal{M}_a$

Let $a: S \to \mathbb{A}$ be a k-scheme.

By technical Lemma, this is equivalent to an arrow

$h_a: X \times S \longrightarrow [\mathfrak{car}/\mathbb{G}_m]$ above $h_D: X\times S \to B\mathbb{G}_m$.

$\rightsquigarrow$ a smooth grp scheme $J_a := h_a^*[J]$ over $X \times S$.

Notation: $\mathcal{P}_a$: Picard cat. of J_a-torsors over $X \times S$

$\mathcal{P}$: the Picard stack over $\mathbb{A}$ given by

$$a \mapsto \mathcal{P}_a.$$

$I_{E,\varphi}$: grp scheme representing sheaf of automorphisms of the pair (E,φ) over $X \times S$

Let $S \in Sch/_k$, let $a \in \mathbb{A}(S)$.

$$\mathcal{M}_a \underset{car}{:=} \{ (E,\varphi) \in \mathcal{M}(S) \text{ "of characteristic } a\text{"} \} =$$

$\mathcal{P}_a \curvearrowright \mathcal{M}_a$?

For $(E,\varphi) \in \mathcal{M}_a$, Recall that we have a canonical morph.

$$\begin{tikzcd}
x^*J \arrow[rr] \arrow[dr] & & I \arrow[r, bend left, "h_a = {[x]} \circ h_{E,\varphi}"] \arrow[dl] & J_a \arrow[r] & I_{(E,\varphi)} \\
& \mathfrak{g} & & &
\end{tikzcd}$$

a grp scheme hom.

$$J_a = h_a^*[J] = ([x] \circ h_{E,\varphi})^*[J] = h_{E,\varphi}^* \circ [x]^*[J]$$

$\rightsquigarrow$ Can twist the pair (E, φ) by any J_a-torsor without changing the characteristic of (E, φ).

$\rightsquigarrow$ $\mathcal{P}_a \curvearrowright \mathcal{M}_a$ • what can we say about this action?

By Chevalley-Kostant restriction theorem,

$\mathrm{car}_{\mathbb{G}} = \mathrm{Spec}\, k[\mathfrak{t}]^W \rightsquigarrow \mathfrak{t} \xrightarrow{\pi} \mathrm{car}_{\mathbb{G}}$ which is finite, generically étale Galois of Galois group W.

$\mathfrak{B}_G :=$ Branch locus of π (In theory of Galois covers, this would mean the closed subscheme of $\mathrm{car}_{\mathbb{G}}$ with ramified fibers)

here $\mathfrak{B}_{\mathbb{G}}$ is the divisor of $\mathrm{car}_{\mathbb{G}}$ defined by the vanishing of the discriminant function $\prod_{\alpha \in \Phi} d\alpha$, $d\alpha : \mathfrak{t} \to \mathbb{G}_a$ is the derivation of the root $\alpha : \mathbb{T} \to \mathbb{G}_m$.

Def: • a characteristic $a \in \mathbb{A}(\bar{k})$ is by def. a section

$$h_a : \bar{X} \longrightarrow \mathrm{car} \times^{\mathbb{G}_m} L_D .$$

• a is said very regular if $h_a(\bar{X})$ meets the divisor $\mathfrak{B} \times^{\mathbb{G}_m} L_D$ transversally (i.e., $h_a(X)$ meets the smooth part of $\mathfrak{B}_{\mathbb{G}} \times^{\mathbb{G}_m} L_D$ with multiplicity 1 on each intersection point).

Rem: For D very ample divisor, the very regular characteristics form a dense open of $\mathbb{A}$ (theorem of Bertini).

Prop 4.3: If $a \in \mathbb{A}(\bar{k})$ is a very regular characteristic

Then $\mathcal{P}_a \circlearrowright \mathcal{M}_a$ is simply transitive.

(in othe words, $\mathcal{M}_a$ is a gerb for the Picard stack $\mathcal{P}$)

Proof: Let $a \in \mathbb{A}(\bar{k})$ be very regular characteristic.

$a \xleftrightarrow{\text{tech lemma}} h_a : \bar{X} \longrightarrow [\mathfrak{car}/\mathbb{G}_m]$ above $h_D : \bar{X} \longrightarrow B\mathbb{G}_m$

$\mathcal{M}_a \ni (E, \varphi)$ S-point $\longleftrightarrow h_{E,\varphi}$ lifting h_a constant on the factor S

$$\begin{tikzcd} X \times S \arrow[r, "h_{E,\varphi}"] \arrow[dr, "h_a"'] & {[\mathfrak{g}/(G \times \mathbb{G}_m)]} \arrow[d, "{[\chi]}"] \\ & {[\mathfrak{car}/\mathbb{G}_m]} \end{tikzcd}$$

(this is just a zoom on the diagram of part 1 of the talk)

a very regular $\Rightarrow h_{E,\varphi}$ factors through the open $[\mathfrak{g}^{reg}/G \times \mathbb{G}_m]$.

But $[\mathfrak{g}^{reg}/G \times \mathbb{G}_m]$ is a $\overbrace{J\text{-gerbe}}^{\text{similar to G-torsor with Picard stack action}}$ (prop 3.4) over $[\mathfrak{car}/\mathbb{G}_m]$

so that $\mathcal{M}_a$ is a $\mathcal{P}_a$-torsor, hence the simply transitive action. $\blacksquare$

Cor: The orbits of $\mathcal{P}_a \circlearrowright \mathcal{M}_a$ are open dense. $\blacksquare$

Let's illustrate this quite abstract construction and relate its objects to what we know in the classical case:

E.g.: [Hitchin, Beauville – Narasimhan – Ramanan]

$G = GL_m$,

$a \rightsquigarrow$ a spectral covering $Y_a \to X$ which is finite of degree m.

$\mathcal{M}_a :=$ compactified Jacobian of $Y_a = \left\{ \begin{array}{l} \text{torsion-free } \mathcal{O}_{Y_a}\text{-module} \\ \text{of generic rank } 1 \end{array} \right\}$

$P_a =$ Jacobian of $Y_a = \{\text{invertible } \mathcal{O}_{Y_a}\text{-modules}\}$

$\mathcal{P}_a \curvearrowright \mathcal{M}_a$ by tensor product.

Rem: when a is no more very regular, $\mathcal{P}_a \curvearrowright \mathcal{M}_a$ is not simply transitive in general. but we still can say something about the quotient stack $[\mathcal{M}_a/\mathcal{P}_a]$ when a is generically semisimple regular.

Def: • A characteristic $a \in \mathcal{A}(\bar{k})$ is generically semisimple regular if the image of the associated $h_a : X \to \mathrm{car} \times^{\mathbb{G}_m} L_D$ is not contained in $\mathfrak{B} \times^{\mathbb{G}_m} L_D$ where $\mathfrak{B} \subset \mathrm{car}$ is the branch locus of $\pi : \mathfrak{t} \to \mathrm{car}$.

• $\mathcal{A}^{\heartsuit} \subset \mathcal{A}$ the open subscheme of $\mathcal{A}$ formed by generically semisimple regular characteristics.

Lemma 4.5 The 2-cat quotient $[\mathcal{M}_a(\bar{k}) / P_a(\bar{k})]$ is equivalent to a 1-category for $a \in \mathcal{A}^{\heartsuit}(\bar{k})$

"proof:" check **Dal criteria**: a 2-cat quotient X by the action of a Picard cat Q is equiv. to a 1-cat iff $\forall x \in \mathrm{ob}(X)$, $\mathrm{Aut}_Q(1_Q) \longrightarrow \mathrm{Aut}_X(x)$ is injective. $\boxed{/\!/}$