# $\infty$-dim VS classical

## GIT

### $\infty$

- Is a "generalization" of classical GIT (HN filtrations, Kempf th ...)
- In the language of "Alg. stacks"
- Avoids Auxilliary choices and the fact that results are indep of choices.
- Doesn't produce proj mod. sp. automatically but gives conditions for existence of the mod. sp.

### classical

- Makes choices

- $\mathcal{F} \in \mathrm{Coh}^d_P(X)$

$\rightsquigarrow$ choice of $m \gg 0$ s.t. $\mathcal{O}_X(-m)^{\oplus P(m)} \overset{q}{\twoheadrightarrow} \mathcal{F}$

$(\mathcal{F}, q) \in Q_m := \mathrm{Quot}(\mathcal{O}_X(-m)^{\oplus P(m)})$

$\rightsquigarrow SL_{P(m)} \curvearrowright Q_m$

$\overset{HM}{\underset{\text{Creterion}}{\rightsquigarrow}}$ GIT stability is indep. of $m$.

- reductive $G \curvearrowright X$ proj.

$$\begin{tikzcd} G \times \mathbb{L} \arrow[r, "\sigma_{\mathbb{L}}"] \arrow[d] & \mathbb{L} \arrow[d] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd}$$

$\rightsquigarrow \gamma \in H^0(X, \mathbb{L})^G$

$\rightsquigarrow X_\gamma := \{x \mid \gamma(x) \neq 0\}$

$\rightsquigarrow X \underset{\text{open}}{\supset} X^{ss}(\mathbb{L}) = \bigcup_\gamma X_\gamma$

- Direct construction of proj. mod. space.

# § 0. Regular principal subschemes & Twists

Def: • $D$ is <u>principal</u> subscheme of $X$, if $\mathcal{I}_D$ is locally principal.

• $D_1, D_2$ principal corresp. to $\mathcal{I}_{D_1}, \mathcal{I}_{D_2}$.

$$D_1 + D_2 \longleftrightarrow \mathcal{I}_{D_1} \cdot \mathcal{I}_{D_2}.$$

• $T$ an $S$-scheme, $\mathcal{F} \in QCoh(T)$

$t \in T$ is associated to $\mathcal{F}$ if $\mathfrak{m}_t$ is <u>associated</u> to the $\mathcal{O}_{T,t}$-module $\mathcal{F}_t$

$Ass_T(\mathcal{F}) :=$ set of associated points of $\mathcal{F}$.

• $T$ is an $S$-scheme, $X_T := X \times_S T$, $D \hookrightarrow X_T$ principal

$\mathcal{F}$ is a $T$-flat finitely presented $\mathcal{O}_{X_T}$-mod.

$D$ is <u>$\mathcal{F}$-regular</u> if $\forall t \in T$, $D_t \cap Ass_{X_t}(\mathcal{F}|_{X_t}) = \phi$

$D_t =$ fiber of $D$ over $t$

$X_t$: —— $X_T$ over $t$.

Lemma 1: $C, D \hookrightarrow X$ principal subschemes.

$\mathcal{F} \in \mathrm{Qcoh}(X)$, $S$-flat.

if $C, D$ are $\mathcal{F}$-regular, then:

a) $C+D$ is regular.

b) $\forall\, T \to S$, $(f_X)^{-1} D$ is $\mathcal{F}|_{X_T}$-regular. $\quad X_T \to X$

c) $\sigma : \mathcal{I}_D \hookrightarrow \mathcal{O}_X$ then:

$\sigma \otimes id_{\mathcal{F}} : \mathcal{I}_D \otimes \mathcal{F} \to \mathcal{F}$ is inj., Moreover.

$\mathcal{F} / \mathcal{I}_D \otimes \mathcal{F}$ is $S$-flat.

d) locally $\mathcal{I}_D \otimes \mathcal{F} \cong \mathcal{F}$.

Proof: a), b) follow from definitions.

c) $\sigma : \mathcal{I}_D \hookrightarrow \mathcal{O}_X$, enough to prove it locally

$\tilde{\mathcal{I}}_D = \langle u \rangle$, $u \in \mathcal{O}_X(X)$, $X$ affine.

exact seq. $\rightsquigarrow$ $\mathrm{Ann}(u) \xhookrightarrow{\alpha} \mathcal{O}_X \xrightarrow{\cdot u} \mathcal{O}_X$

$\rightsquigarrow \operatorname{coker} \alpha \cong \mathcal{O}_X / \operatorname{Im}(\alpha) \cong \mathcal{O}_X / \ker(\cdot u) \cong \operatorname{Im}(\cdot u) = I_D$

$\otimes \tilde{\mathcal{F}} \rightsquigarrow$

$$\begin{tikzcd} \operatorname{Ann}(\mathcal{F}) \otimes \tilde{\mathcal{F}} \arrow[r, "\alpha \times \mathrm{id}_{\tilde{\mathcal{F}}}"] & \tilde{\mathcal{F}} \arrow[r, "\cdot u \times \mathrm{id}_{\tilde{\mathcal{F}}}"] \arrow[d] & \mathcal{F} \\ & I_D \cong \operatorname{coker}(\alpha \times \mathrm{id}_{\mathcal{F}}) \arrow[ur, dotted] & \end{tikzcd}$$

$\mathcal{F}$ is $D$-regular $\rightsquigarrow$ $\forall s \in S$, $\mathcal{F}|_{X_s} \xrightarrow{(\cdot u \otimes \mathrm{id}\mathcal{F})|_{X_s}} \mathcal{F}|_{X_s}$

slicing criteria $\rightsquigarrow$ for flatness [Tag 00ME]

$\tilde{\mathcal{F}}$ is $S$-flat $\Rightarrow \begin{cases} \tilde{\mathcal{F}} \xrightarrow{\cdot u \times \mathrm{id}\mathcal{F}} \mathcal{F} \text{ is inj} \\ \mathcal{F}/_{u, \mathcal{F}} \text{ is } S\text{-flat.} \end{cases}$

d) locally c) $\Longrightarrow I_D \otimes \mathcal{F} \cong \operatorname{coker}(\alpha \otimes \mathrm{id}_{\mathcal{F}})$

$C \Longrightarrow \cdot u \times \mathrm{id}_{\mathcal{F}}$ is inj.

$\rightsquigarrow \operatorname{Im}(\alpha \otimes \mathrm{id}_{\mathcal{F}}) \subset \ker(\cdot u \otimes \mathrm{id}_{\mathcal{F}}) = 0$

then $I_D \otimes \mathcal{F} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} / 0 \cong \mathcal{F}$. $\square$

Rem: $\tilde{\mathcal{F}} \longrightarrow j_* j^* \mathcal{F}$ is inj.

$j^* \tilde{\mathcal{F}} \longrightarrow j_* j^* \mathcal{F}$ is injective

$\mathcal{F} \longrightarrow j^* \mathcal{F}$ is inj $\hookrightarrow$ restriction of scalars along $j^\#$, $(\cdot u)$.

$j : U = X \setminus D \longrightarrow X$

$\quad\; \|$

$\quad D(u)$

$j^\# : A \longrightarrow A_u$.

$(\cdot u)$ is injective from Lemma 1 (c).

Def: (Twists). Consider $\mathcal{I}_D \otimes j_* j^* \mathcal{F} \xrightarrow{\alpha_u} j_* j^* \mathcal{F}$ (Lemma 1) d))

- $\mathcal{F}(-D) := \alpha_u(\mathcal{I}_D \otimes \tilde{\mathcal{F}})$
- $\mathcal{F}(D) := \max\{\mathcal{E} \subset j_* j^* \mathcal{F} \mid \alpha_u(\mathcal{I}_D \otimes \mathcal{E}) \subset \mathcal{F} \subset j_* j^* \mathcal{F}\}$

$\mathcal{F}(-D) := \alpha_u(\mathcal{I}_D \otimes \mathcal{F}) \underset{d)}{\overset{\text{lemma 1}}{\simeq}} \mathcal{F}$ so $\mathcal{F}(-D) \hookrightarrow \mathcal{F}$.

$\mathcal{F} \subset j_* j^* \mathcal{F}$ , $\alpha_u(\mathcal{I}_{\sim D} \otimes \mathcal{F}) \cong \mathcal{F} \hookrightarrow j_* j^* \mathcal{F}$

$\boxed{\mathcal{F} \hookrightarrow \mathcal{F}(D)}$

$\rightsquigarrow \cdots \mathcal{F}(-mD) \hookrightarrow \cdots \hookrightarrow \mathcal{F}(-D) \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{F}(D) \hookrightarrow \cdots$

$\mathcal{F}(mD) \hookrightarrow \cdots$

local interpretation $\rightsquigarrow$ of $j^*$

$$j_* j^* \mathcal{F} = \operatorname*{colim}_{m \in \mathbb{Z}} \mathcal{F}(mD)$$

<u>Rem:</u> $\mathcal{F}(mD) := \alpha_u(\mathcal{I}_{mD} \otimes \mathcal{F}) \overset{\text{lemma 1}}{\underset{d)}{\cong}} \mathcal{F}$ locally $\forall m \in \mathbb{Z}$.

In particular, $\mathcal{F}$ is $S$-pure of dim $d \implies$ So is $\mathcal{F}(mD)$ $\forall m \in \mathbb{Z}$.

## §1. <u>Stacks of Rational maps:</u>

- $\pi : Y \to T$ morph of schemes of finite type.

$\mathcal{F} \in \mathrm{Qcoh}(Y)$.

$\mathcal{F}$ is $T$-pure of dimension $d$ if $\mathcal{F}$ is $T$-flat, finitely presented and $\forall t \in T$, $\mathcal{F}_t$ is pure of dim. $d$.

$\mathcal{F}_t = \mathcal{F}|_{Y_t}$ , $Y_t$ = fiber of $Y$ over $T$.

- $\mathrm{Coh}^d(X) : (Sch/_S)^{op} \longrightarrow$ Groupoids.

$$T \longmapsto \left\{ \begin{array}{l} \text{Groupoids of } T\text{-pure } \mathcal{O}_{X_T}\text{-mod} \\ \text{of dim } d \end{array} \right\}$$

- $\mathrm{Coh}^d(X)_{rat} : (Aff\, Sch/_S)^{op} \longrightarrow$ Cat

$$T \longmapsto \left\{ (D, \mathcal{E}) \;\middle|\; \begin{array}{l} (1)\ \mathcal{E} \text{ is a } T\text{-pure sheaf of dimension } d \text{ on } X_T. \\ (2)\ D \hookrightarrow X_T \text{ is an } \mathcal{E}\text{-reg. principal subscheme of } X_T \end{array} \right\}$$

$A = (D_1, \mathcal{E}_1)$, $B = (D_2, \mathcal{E}_2) \in \mathrm{Coh}^d(X)_{rat}(T)$

$$\mathrm{Morph}_{\mathrm{Coh}^d(X)_{rat}}(A, B) = \left\{ (i, \psi) \;\middle|\; \begin{array}{l} (1):\ i : D_1 \hookrightarrow D_2 \text{ inclusion} \\ (2)\ \psi : \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \text{ a mono that is an iso } \psi|_{X \setminus D_2} \end{array} \right\}$$

<u>Rem</u>: $\exists\, \eta : \mathrm{Coh}^d(X) \longrightarrow \mathrm{Coh}^d(X)_{rat}$

$$\mathcal{E} \longmapsto (\phi, \mathcal{E})$$

# §2. Affine Grassmannians of pure sheaves

Comma cat.

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

$(c, e, \varphi : F(c) \to G(e))$

natural transf.

$$F, D : \mathcal{E} \xrightarrow{\eta} \mathcal{D}$$

$$\eta(T) : \mathcal{E}(T) \xrightarrow{\eta(T)} \mathcal{D}(T)$$

<u>Def:</u> $(D, \mathcal{E})$ an $S$-point in $Coh^d(X)_{rat}$.

Affine Grassmannian

$$Gr_{X,D,\mathcal{E}} : \quad Coh^d(X) \underset{Coh^d(X)_{rat}}{\times} S$$

$$\begin{tikzcd} & Coh^d(X) \arrow[d, "\alpha"] & \mathcal{F} \arrow[d] \\ S \arrow[r, "\beta"'] & Coh^d(X)_{rat} & (\phi, \mathcal{F}) \arrow[dl] \\ T \arrow[r, mapsto] & (D, \mathcal{E}) & (i, \psi) \end{tikzcd}$$

$i : \Phi \hookrightarrow D$ ✓

$\Psi$ a momo s.t $\Psi|_{X_T \setminus D}$ is an iso.

$$(\mathcal{F}, \cancel{T}, (\cancel{i}, \Psi))$$

$$Gr_{X,D,\mathcal{E}}(T) = \left\{ (\mathcal{F}, \Psi) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is } T\text{-para of dim } d \text{ s.t.} \\ D_T \text{ is } \mathcal{F}\text{-reg} \\ \Psi : \mathcal{E}_T \to \mathcal{F} \text{ momo s.t.} \\ \Psi|_{U_T} \text{ is an iso} \end{array} \right\}$$

- $(\mathcal{F}_1, \Psi_1) \overset{\Phi}{\cong} (\mathcal{F}_2, \Psi_2)$ is an iso $\mathcal{F}_1 \cong \mathcal{F}_2$ s.t. $\Phi \circ \Psi_1 = \Psi_2$.

$$\begin{tikzcd} \mathcal{F}_1 \arrow[r, "\Phi"] & \mathcal{F}_2 \\ \mathcal{E}_T \arrow[u, "\Psi_1"] \arrow[r, equal] & \mathcal{E}_T \arrow[u, "\Psi_2"'] \end{tikzcd}$$

**Prop:** $Gr_{X,D,\mathcal{E}}$ is represented by a strict ind-scheme that is ind-proj over $S$.

**Def:** $N \in \mathbb{Z}_{\geq 0}$.

$$Gr^{\leq N}_{X,D,\mathcal{E}}(T) = \left\{ \begin{array}{c} (\mathcal{F},\psi) \text{ in } Gr_{X,D,\mathcal{E}}(T) \text{ s.t.} \\ \mathcal{E}_T \subset \mathcal{F} \subset \mathcal{E}_T(ND_T) \end{array} \right\}$$

**Rem:** $\forall N \leq M$, $\mathcal{E}_T(ND_T) \hookrightarrow \mathcal{E}_T(MD_T)$

$\rightsquigarrow$ Natural inclusions $Gr^{\leq N}_{X,D,\mathcal{E}} \subset Gr^{\leq M}_{X,D,\mathcal{E}}$

**Lemma 2:** $Gr_{X,D,\mathcal{E}} = \varinjlim_{N \geq 0} Gr^{\leq N}_{X,D,\mathcal{E}}$ as presheaves on $\mathrm{Affsch}/S$.

**Proof:** $T \in \mathrm{Affsch}/S$, let $(\mathcal{F},\psi) \in Gr_{X,D,\mathcal{E}}(T)$

$\exists ?\ N \geq 0$ : $(\mathcal{F},\psi) \in Gr^{\leq N}_{X,D,\mathcal{E}}(T)$.

$X \xrightarrow{f} S$ is of finite pres.

$X_T \xrightarrow{f_T} T$ ——————, $\rightsquigarrow$ l.o.p, q-sep, q-compact.

$T$ aff $\rightsquigarrow$ $b_T^{-1}(T) = X_T$ is q-compact

$\rightsquigarrow$ assume $X_T$ is affine, $D_T$ cut out by $x \in \mathcal{O}_{X_T}$

$\rightsquigarrow$ $\mathcal{F}$ and $\mathcal{E}_T$ are fini presented. without $x$-torsion.

$\rightsquigarrow$ choose a set of generators of $\mathcal{F}$.

$(e_i)_{i \in I}$ , $|I| < \infty$.

By assumption $\mathcal{F}[\frac{1}{x}] = \mathcal{E}_T[\frac{1}{x}]$

So $\forall i \in I$, $\exists m_i \in \mathbb{Z}_{>0}$ s.t. $x^{m_i} e_i \in \mathcal{E}_T$

$\rightsquigarrow$ $N := \max_{i \in I} \{m_i\}$ $\rightsquigarrow$ $\mathcal{F} \subset x^{-N} \mathcal{E}_T = \mathcal{E}_T(N D_T)$

<u>Lemma 3:</u> $\forall N \in \mathbb{Z}_{>0}$, $Gr^{\leq N}_{X,D,\mathcal{E}}$ is repres. by a disj. union of schemes that are proj. of finite pres. over $S$.

Moreover, $\forall N \leq M$, $Gr^{\leq N}_{X,D,\mathcal{E}} \hookrightarrow Gr^{\leq M}_{X,D,\mathcal{E}}$

Proof: $\forall N \in \mathbb{Z}_{>0}$, $Q^N_{X,D,\mathcal{E}} := \mathrm{Quot}_{X/S}\left(\mathcal{E}(ND)/\mathcal{E}\right)$

So $Q^N_{X,D,\mathcal{E}}(T) = \left\{ (\mathcal{F}, q) \;\middle|\; \begin{array}{l} \mathcal{F}: T\text{-flat on } X_T \\ \mathcal{E}_T(ND_T)/\mathcal{E}_T \xrightarrow{q} \mathcal{F} \end{array} \right\}$

Mistake (2nd line of Proof of Lemma 3.16) p 21.

Idea:

$$\begin{tikzcd} \mathrm{Gr}^{\leq N+1}_{X,D,\mathcal{E}} \arrow[r, "\phi_{N+1}", "\cong"'] & Q^{N+1}_{X,D,\mathcal{E}} \\ \mathrm{Gr}^{N}_{X,D,\mathcal{E}} \arrow[u, hook, "g_N"] \arrow[r, "\phi_N", "\cong"'] \arrow[ur, phantom, "\circlearrowleft"] & Q^{N}_{X,D,\mathcal{E}} \arrow[u, hook, "t_N"'] \end{tikzcd} \quad (\star)$$

i) construct $t_N$ a closed immersion

ii) ——— $\phi_N$

iii) ——— $t_N$

~~iv)~~ $\phi_N = t_N^{-1}$

~~v)~~ $\star$ commutes.

i) $t_N : Q^N_{X,D,\mathcal{E}} \longrightarrow Q^{N+1}_{X,D,\mathcal{E}}$

$$T \longmapsto \frac{\mathcal{E}_T((N+1)D_T)}{\ker}$$