Talk 5. wall + chamber structure

Recall: stab(X): coarsest topology that makes:

$(\mathcal{A}, Z) \mapsto Z$ and $(\mathcal{A}, Z) \mapsto \phi^{\pm}(Z(E))$ $\forall E \in \mathrm{Coh}(X)$

both continuous functions.

Change of stable objects ⇝ change stab.

Aim: make precise and establish what is a wall on

stab(X).

Def: Let $v_0, w \in \Lambda \setminus \{0\}$ be non-parallel.

A numerical wall $W_w(v_0)$ for v_0 w.r.t. w

$$W_w(v_0) := \{ \underset{\substack{\cap \\ \mathrm{stab}(X)}}{\sigma = (\mathcal{P}, Z)} \mid \mathrm{Re}(Z v_0)\,\mathrm{Im}(Z w) = \mathrm{Re}(Z w)\cdot \mathrm{Im}(Z v_0) \}$$

↳ stabs that agree with $Z(v_0) \sim Z(w)$

The set of such walls will be $W(v_0)$

We will consider walls w.r.t. $S \subset \mathrm{stab}(X)$. It will be denoted by $W^S(v_0)$

<u>Proposition:</u> Let $v_0 \in \Lambda$ be a primitive class and let $S \subset D^b(X)$ of objects of class v_0.

Then $\exists$ collection of walls $W^S_w(v_0)$ with:

(1) Every wall in $W^S_w(v_0)$ is a closed submanif with bound of real codim $= 1$

(2) The collection is locally finite.

(3) $\forall\ \sigma = (\mathcal{P}, Z)$ on $W^S_w(v_0)$, $\exists\, \phi \in \mathbb{R}$ and $\exists\ F_w \hookrightarrow E_{v_0}$ in $\mathcal{P}(\phi)$ with $v(F_w)$ and

$E_{v_0} \in S$

(4) If $\underset{\text{subset}}{c \subset C}$ $\operatorname{stab}(X)$ which is a chamber, and $\sigma_1, \sigma_2 \in C$ then

E_{v_0} is σ_1-stable iff E_{v_0} is σ_2-stable.

Proof: Let V_w^S be the set of stabs for which (3) is true.

Since v_0 is primitive, this subset is contained in $\operatorname{Im} \frac{Z v_0}{Z w} = 0$

Let us show that $\exists$ only finitely many classes w s.t. V_w^S intersects

$B_{1/8}(\sigma)$, $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}(X)$.

Let $I_\sigma(S) \subset \Lambda$ be the set of w for which

$\exists\, \phi \in \mathbb{R}$ with $Z(v_0) \in \mathbb{R}_{>0} e^{i\pi\phi}$ and $\exists$ a strict

inclusion $F_w \hookrightarrow E$ in $P(\phi - \frac{1}{4}, \phi + \frac{1}{4})$ w/ $v(F_w) = w$

and $E \in S$.

Since $\|\cdot\|_\sigma$ is finite and since Λ is discrete then $\exists$ only finite

$\gamma \in \Lambda$ can satisfy $\exists\, F_\gamma$ σ-ss of class $v(F_\gamma) = \gamma$

and $|Z F_\gamma| < |Z v_0|$

It follows that $I_\sigma(S)$ is also finite (since HN-filt. are

obtained as objects above). But if V_w^S intersects $B_{1/8}(\sigma)$

Then $w \in I_\sigma(S)$ $\left(\sigma' \in V_w^S \cap B_{1/8}(\sigma)\right)$

$\exists\, F_w \hookrightarrow E_{v_0} \in \mathcal{P}(\phi)$ [Lem. 7.5 Stab. cond. Bridgeland]

An obj E of class v_0 is $(\mathcal{P}', Z')$-stable for $(\mathcal{P}', Z') \in B_{1/8}(\sigma)$ iff

$$\operatorname{Im} \frac{Z' w}{Z' v_0} \leq 0 \text{ for every } w \in I_\sigma(\{E\})$$

Repeating this argument for all subobjects F_w it follows that inside the codim 1 subset $\operatorname{Im} \frac{Z' w}{Z' v_0} = 0$, the set V_w^S is a finite union of subsets each of which is cut by an inequality $\frac{\operatorname{Im} Z'(w')}{Z'(v_0)} \leq 0$ for some $w' \in I_\sigma(S)$.

Let W_w^S be the union of codim 1 components of V_w^S.

So only remains to show condition (4)

For (4): Consider $\sigma_1, \sigma_2 \in B_{1/8}(\sigma) \cap C$

Assume $\exists\ E \in$ [illegible] that is σ_1-st but not σ_2-st.

Then $\gamma : [0,1] \longrightarrow B_{1/8} \cap C$ connects σ_1 and σ_2,

$\exists\ t \in [0,1]$ s.t. E is strictly $\gamma(t)$-ss i.e. $\gamma(t) \in V_w^S \cap C$ for some $w \in I_\sigma(S)$. By def. of walls the set $V_w^S \cap C$ has codim at least 2

We γ to avoid all nonempty compos of V^S_w when $t \in (0,1)$

In particular σ_2 is $\underbrace{\text{contained}}_{\text{as } E \text{ stri. ss}}$ in $\frac{Z v_0}{Z w} = 0$ and E will not be stable

in $B_{1/8}(\sigma) \cap C$ with $\operatorname{Im} \frac{Z v_0}{Z w} \leq 0$

But since $C \setminus \sqcup V^S_w$ is path connected and by repeating the same

$\underbrace{\text{argument}}_{\text{pushing the unstable } E \text{ until the end of the path.}}$ we have that E is stable in all of $C \setminus \sqcup V^S_w$

This is $\Leftrightarrow$ to $\exists$ of stab. that thinks that thinks that E is strictly ss. $\blacksquare$

How walls behave w.r.t. the quadratic form from the support condition.

<u>Recall:</u> $C_\sigma := \inf \left\{ \frac{|Z(v(E))|}{\|v(E)\|} \;\middle|\; \forall\, 0 \neq E \in \mathcal{P}(\phi),\ \phi \in \mathbb{R} \right\}$

Define the quadratic form: $Q(w) = C_\sigma^2 |Z(w)|^2 - \|w\|^2$

It has the following properties:

(1) all E ss $E \in \mathcal{P}$ satisfy $Q(v(E)) \geq 0$

(2) $v \neq 0 \in \Lambda_{\mathbb{R}}$ with $Z(v) = 0$ then $Q(v) < 0$

Rem: $Q(v(E)) \geq 0$

$\sim$ Bogomolov's

(5.29 macri)

Lemma: let Q be a quadratic form on V as above

$(\exists Z : V \to \mathbb{C}$ and $Q < 0$ on $\ker Z)$

If P is a ray in $\mathbb{C}$, let:

$$C_P^+ = Z^{-1}(P) \cap \{Q \geq 0\}$$ and we have

(1) if $w_1, w_2 \in C_P^+$ then $Q(w_1, w_2) \geq 0$

(2) the set C_P^+ is a convex cone

(3) let $w, w_1, w_2 \in C_P^+$ w/ $w = w_1 + w_2$ then

$0 \leq Q(w_1) + Q(w_2) \leq Q(w)$ moreover

$Q w_1 = Q(w) \Rightarrow Q(w_1) + Q(w_2$

$\vdots$

check macri Lem. 5.29

Proof: for any $0 \neq w_1, w_2 \in C_P^+$, $\exists \lambda \in \mathbb{R}$ s.t.

$Z(w_1 - \lambda w_2) = 0$. Therefore we get:

$0 \geq Q(w_1 - \lambda w_2) = Q(w_1) + \lambda^2 Q(w_2) - 2\lambda Q(w_1 \cdot w_2)$

The ineq. $Q(w_1) \geq 0$ and $Q(w_2) \geq 0 \Rightarrow Q(w_1, w_2) = 0$

Therefore $0 = 2\lambda Q(w_1, w_2) \geq 2 w_1 + \lambda^2 Q w_2$

$$\begin{array}{c} \| \\ Q w_1 \geq 0 \end{array}$$

Let $w \in C_p^+$ w/ $Q(w) = 0$.

Assume $w = w_1 + w_2$ as before, $Z(w_1 - \lambda w_2) = 0$

$0 > Q(w_1 - \lambda w_2) = Q w_1 + \lambda^2 Q w_2 - Q(\lambda^2 w_1 w_2) = 0$

$\blacksquare$

Cor: Assume U path connected open subset of $\mathrm{Stab}(x)$ s.t. all $\sigma \in U$ s.t. the supp. cond w.r.t. Q.

If $E \in D^b(x)$ with $Q(E) = 0$ and E σ-st for some $\sigma \in U$ then E is σ'-st. $\forall \sigma' \in U$.

Proof: If $\exists$ point in U in which E is unstable then $\exists \sigma$ s.t. E is strictly ss, If P is the ray to which $Z(E)$ belongs then by the previous lemma, P is an extremal ray. This is a contrad. to E being strictly ss because if E_i are the Jordan-Holder factors of E then $v(E_i)$ lie on the same extremal face but this is impossible since all near stability conditions will push $v(E_i)$ to $Q > 0$ which in turn turns E into a stable object.