

I) Universal spectral data morph.

II) Weyl's polarisation

III) Spectral data morphism and Hitchin map

- Setup: - X a proper smooth alg. var. of dim d over k .
(we're no more restricted to dim 1)

- \mathcal{T}_X : tangent sheaf of X , Ω_X^1 : sheaf of 1-forms on X

- G : Split reductive grp over k of rank n , $\text{Lie}(G) = \mathfrak{g}$

I) Universal spectral data morph.

Def: A G -Higgs bundle over X is a pair (E, θ) s.t.

1) $E \rightarrow X$ is a G -bundle.

2) $\theta \in H^0(X, \text{ad } E \otimes \Omega_X^1)$ seen as an \mathcal{O}_X -lin map $\mathcal{T}_X \xrightarrow{\theta} \text{ad}(E)$

$$\left(\text{ad } E \otimes_{\mathcal{O}_X} \Omega_X^1 \cong \text{ad } E \otimes \mathcal{T}_X^* \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, \text{ad } E) \right)$$

s.t. $\forall v_1, v_2$ sections of \mathcal{T}_X $[\theta(v_1), \theta(v_2)] = 0$. (integrability condition)

Def: the commuting scheme $\mathcal{C}_G^d \subset \mathfrak{g}^d$ is the scheme

theoretic zero fibre of the commutator map

$$\begin{aligned} \mathfrak{g}^d &\longrightarrow \prod_{i < j} \mathfrak{g} \\ (\theta_1, \dots, \theta_d) &\longmapsto \prod_{i < j} [\theta_i, \theta_j] \end{aligned}$$

$$\mathcal{E}_G^d(k) = \{ (\theta_1, \dots, \theta_d) \in \mathfrak{g}^d(k) \text{ s.t. } [\theta_i, \theta_j] = 0, 1 \leq i, j \leq d. \}$$

The k -points

note that the commuting relations are automatically satisfied for $d=1$.
we seek an alternative description of \mathcal{E}_G^d

k^d equipped with std basis v_1, \dots, v_d , $V_d := (k^d)^V$

$$\mathfrak{g}^d \xrightarrow{\sim} (V_d)^V \otimes \mathfrak{g}$$

$(\theta_1, \dots, \theta_d) \mapsto \theta: V_d \rightarrow \mathfrak{g}$ k -linear s.t. $\theta(v_i) = \theta_i$ (θ is unique)

$\mathcal{E}_G^d :=$ closed subscheme of \mathfrak{g}^d consisting of k -lin. maps

$\theta: V_d \rightarrow \mathfrak{g}$ s.t. $[\theta(v), \theta(v')] = 0 \quad \forall v, v' \in V_d$

$GL_d \times G \curvearrowright \mathcal{E}_G^d$ (coming from $GL_d \curvearrowright V_d$, $G \curvearrowright \mathfrak{g}$)
natural adjoint

quotient stack $[\mathcal{E}_G^d / (GL_d \times G)] := \mathcal{H}$ called "Higgs stack"

it sends a test scheme S to the groupoid of triples

$$\mathcal{H}: S \mapsto \left\{ (V, \mathcal{E}, \theta) \mid \begin{array}{l} V \text{ is a } k\text{-d V.B. over } X \times S \\ \mathcal{E}: \text{ a principal } G\text{-bundle over } X \times S \\ \theta: V \rightarrow \text{ad}(\mathcal{E}) \text{ s.t. } [\theta(v), \theta(v')] = 0 \\ \text{(} G\text{-linear)} \\ \forall v, v' \text{ local sections of } V \end{array} \right\}$$

using technical lemma from last talk, A Higgs field can be represented by θ lying over $X \rightarrow [BG]_d$

cot. bundle $\{ T_x^* \}$ $\left[\mathbb{C}^d / (G_d \times G) \right]$ $T_x^* \rightsquigarrow GL_d$ -torsor

$$X \longrightarrow [BGL_d]$$

Recall: In dim 1 (last talk), we used Chevalley Kostant restriction theorem to derive a morphism of quotient stacks.

$$\left[\mathfrak{g} / G \times G_m \right] \xrightarrow{[x]} \left[\text{car} / G_m \right] \quad \text{for } G \overset{\text{adj}}{\curvearrowright} \mathfrak{g}, G_m \overset{\text{homoth.}}{\curvearrowright} \mathfrak{g}, \text{car} = \mathfrak{g} // G$$

to construct the Hitchin morphism in the language of stacks

$$\rightsquigarrow h_x: \mathcal{M}_x \longrightarrow \mathbb{A}_x := H^0(x, \text{car} \times L_D^{G_m})$$

$$\underbrace{\mathcal{M}_x}_{\text{called } H \text{ in last talk}} : \left(\text{Sch} / k \right)^{\text{op}} \longrightarrow \text{cat}$$

$$S \longmapsto \left\{ \begin{array}{l} h_a: X \times S \longrightarrow \mathfrak{g} / G \times G_m \\ \text{above } h_D: X \times S \longrightarrow [B G_m] \end{array} \right\}$$

(unfortunate)

In higher dim, it's similar,

construction of the Hitchin map derives from G -inv. functions on \mathbb{C}^d / G as we will see with Weyl's polarisation construction, so studying the Hitchin map amounts to studying

$$\mathbb{C}^d // G = \text{Spec} \left(k \left[\mathbb{C}^d / G \right]^G \right) \text{ GIT quotient, } G \overset{\text{diag}}{\curvearrowright} \mathbb{C}^d \overset{\text{adj}}{\curvearrowright} G$$

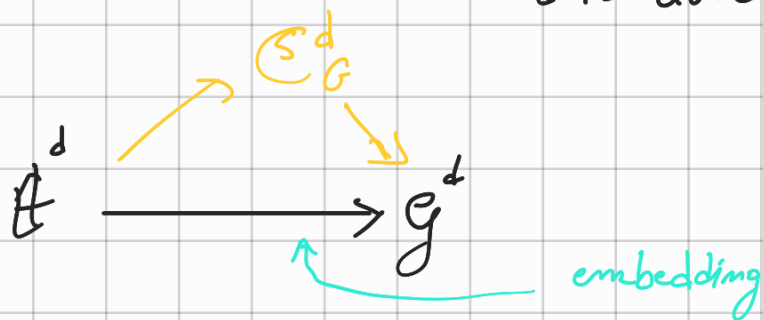
$[\mathbb{A}_G^d / G]$ = quotient stack for $G \curvearrowright^{\text{diag}} \mathbb{A}_G^d$

$$[\mathbb{A}_G^d / G] \xrightarrow{q} \mathbb{A}_G^d // G$$

Now $\mathcal{M}_x: (\text{Sch}/k)^{\text{op}} \rightarrow \text{cat}$
 in higher dim
 $S \mapsto \left\{ \begin{array}{l} \vartheta: X \times S \rightarrow [\mathbb{A}_G^d / (G \times G^L_d)] \\ \text{above } X \times S \rightarrow [\text{BGL}_d] \end{array} \right\}$

let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra (in particular \mathfrak{h} is abelian)

\leadsto a factorisation



orbits $(W \curvearrowright^{\text{diag}} \mathfrak{h}^d) \subset \text{orbits } (G \curvearrowright^{\text{diag}} \mathbb{A}_G^d)$

$$\leadsto \forall \mathcal{G} \in k[\mathbb{A}_G^d]^G, \mathcal{G}|_{\mathfrak{h}^d} \in k[\mathfrak{h}^d]^W$$

$$\leadsto \mathfrak{h}^d // W \xrightarrow{\alpha} \mathbb{A}_G^d // G \quad (\text{corresponding morph of off. schemes})$$

Humziker '97 : α is a universal homeomorphism

(i.e., finite morphism inducing a bijection on k points)

In particular, $\mathfrak{h}^d // W$ is the normalisation of $(\mathbb{A}_G^d // G)^{\text{red}}$

(the underlying reduced subscheme)

conjecture 1 [Chen - Ngô '20] : α is an iso.

(equivalently, $\mathbb{A}_G^d // G$ is reduced and normal)

Rem:

1) conjecture 1 $\iff \mathbb{A}_G^d // G$ is reduced + normal. Indeed

easy part \implies if α is an iso, then

$\mathbb{A}^d // W$ reduced + normal $\implies \mathbb{A}_G^d$ reduced + normal

\Leftarrow) if $\mathbb{A}_G^d // G$ reduced + normal, then

α is a normalisation (by Hurziker '97) $\implies \alpha$ an iso

2) there is a long-standing conjecture saying that the

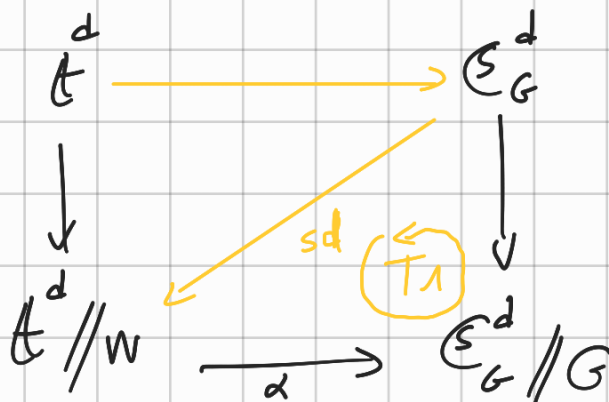
scheme \mathbb{A}_G^2 is reduced. For $d \geq 3$ it seems to be doubtful.

But in general $\mathbb{A}^d // G$ behaves better as we will see.

Def: we call a universal spectral data morphism a G -inv map sd

s.t.

(making the following diagram commute)



note that

sd is completely

indep. of X

which is amazing

Rem: 1) \exists of S_d is always satisfied by conjecture 1 + the morph

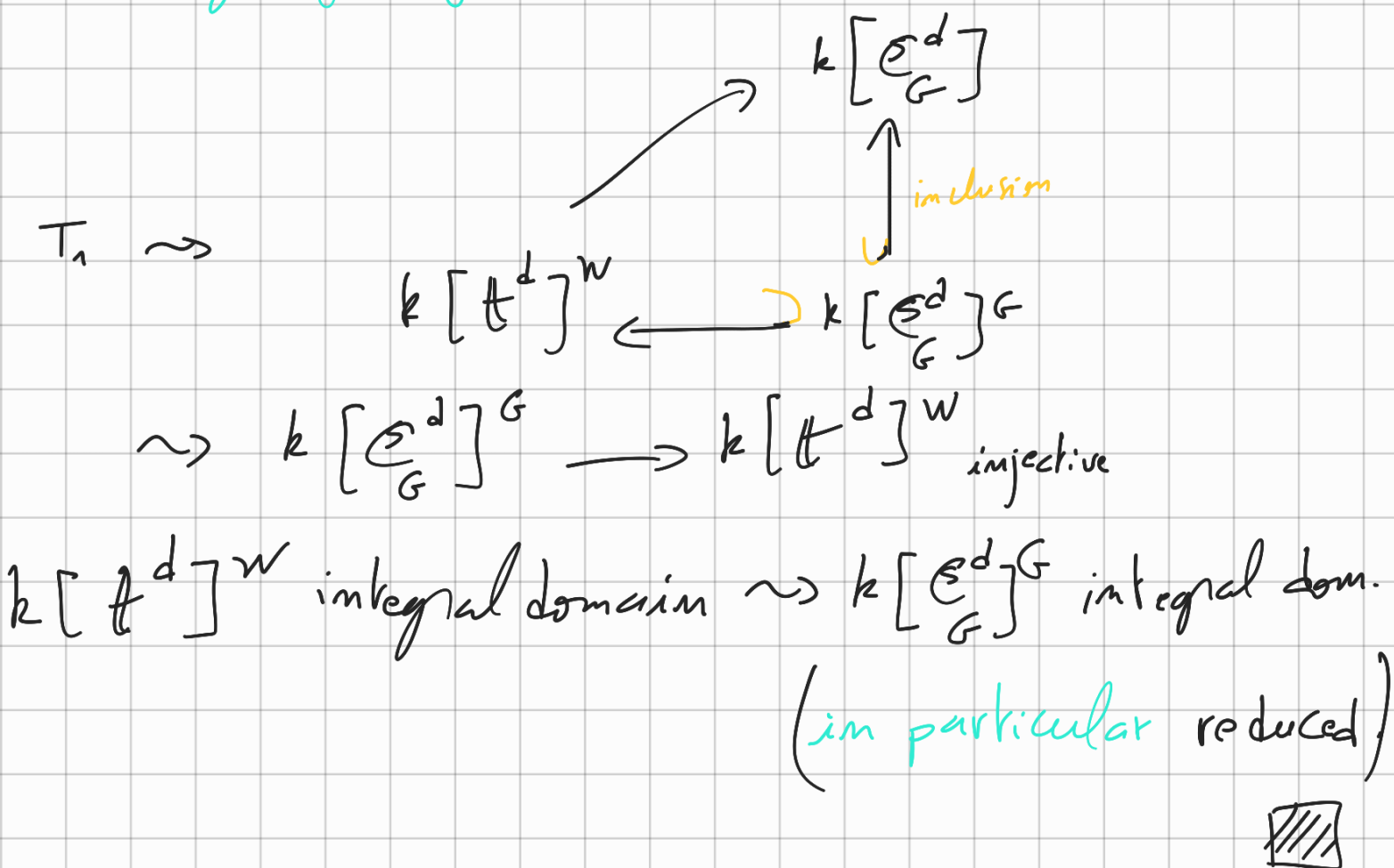
$$\left[\mathbb{C}_G^d / G \right] \xrightarrow{q} \mathbb{C}_G^d // G$$

since we don't have a proof of conjecture 3.1. and that this conjecture is crucial for the study of the Hitchin map, we state a weaker conjecture.

Conjecture 2 [Chen-NGO '20]: \exists a G -inv map $sd.$ s.t. $(+)$ commutes.

Rem: conjecture 2 $\Rightarrow \mathbb{C}_G^d // G$ is reduced.

Indeed, the right triangle of $(+)$ gives a commutative triangle of rings:



Th [Deligne] conjecture 2 holds for $G = GL_n$.

In particular, $\mathbb{C}_G^d // G$ is reduced.

proof idea: construct S_d using the well-known fact

that $k[\mathfrak{g}^d]^G$ is generated by $T_r(x_{i_1} \dots x_{i_k})$

where $k \in \mathbb{Z}_{\geq 0}$, $1 \leq i_1, \dots, i_k \leq d$

II) Weyl's polarisation

Roughly speaking Weyl's polarisation is a way to construct G -inv. functions on the space \mathfrak{g}^d of d -tuples in \mathfrak{g} .

Given $c \in k[\mathfrak{g}]^G$, $x_1, \dots, x_d \in k$, and

$$\mathfrak{g}^d \xrightarrow{\psi} k$$

let $(\theta_1, \dots, \theta_d) \mapsto c(\theta_1 + \dots + \theta_d)$

Let $\text{pol}_d k[\mathfrak{g}]^G$ be the subalgebra of $k[\mathfrak{g}^d]^G$ generated

by all the ψ 's.

Q: when does polarisations of G -inv. functions of \mathfrak{g} generate the alg.

of G -inv. funcs on \mathfrak{g}^d : $\text{pol}_d[V]^G = k[V^d]^G$ hold? and this is a

classical problem of invariant theory when you replace \mathfrak{g} by V a

general fin. dim. alg. G -module.

A: It depends, sometimes

• It does for instance for $G = \underbrace{O_m}_{\text{mat. action}} \curvearrowright V = k^m$ (th. of Study)

$G = \underbrace{S_m}_{\text{permute coords}} \curvearrowright V = k^m$ (th. of Weyl)

• It doesn't in general, for instance,

$$G = \text{SL}_m^{\text{mat}} \curvearrowright V = k^m$$

$$\leadsto k[V]^G = k \leadsto \text{pol}_m k[V]^G = k \neq k[V^m]^G$$

Nevertheless polarisations are close to generate $k[V]^G$ as we have this result:

[Hunziker '97, Cor. 2.16]: let G be finite,

let V be a fin. dim. alg. G -module, $m \in \mathbb{N}$.

Then $k[V^m]^G$ is the integral closure of $\text{pol}_m k[V]^G$ in $k(V^m)^G$.

Let's formalize Weyl polarisation construction for some aff. alg. var.

For an affine variety Y with $G_m \curvearrowright Y$,

$$F: R \in \text{Alg}_k \longmapsto \left\{ G_m\text{-equiv map } V_d \otimes_k R \longrightarrow Y \right\}$$

F is representable by $Y_{G_m}^{V_d} \in \text{AffSch}$

E.g.:

$$1) Y := \mathbb{A}_k^1, \quad G_m \curvearrowright Y \text{ via: } t \cdot x = t^e x$$

Then $Y_{G_m}^{V_d} = e\text{-th symmetric tensor } \text{Spec}(S^e S(V_d))$
by univ. prop. of symm. alg.

$$2) Y := \mathfrak{g} \quad \text{then } Y_{G_m}^{V_d} \text{ can be identified with } \mathfrak{g}^d$$

3) $\gamma := \mathbb{C} = \mathfrak{g} // G$ $\mathbb{C} \cong$ m -dim aff. space with homog. coord.
 Chevalley restriction C_1, \dots, C_m of degrees e_1, \dots, e_m

So $\mathbb{C}_{G_m}^{\vee d} \cong \prod_{i=1}^m S^{e_i} A^d =: A$ (this iso depends on the choice of the homog. coordinates)

Since $\mathfrak{g} \rightarrow \mathfrak{g} // G$ is G -invariant and G_m -equiv.,
 (for $G \curvearrowright_{\text{adj}} \mathfrak{g}$)

induces $\text{pol}: \mathfrak{g}^d \rightarrow A$ (G -invariant for $G \curvearrowright_{\text{diag}} \mathfrak{g}^d$)
 "Weyl's polarisation construction"

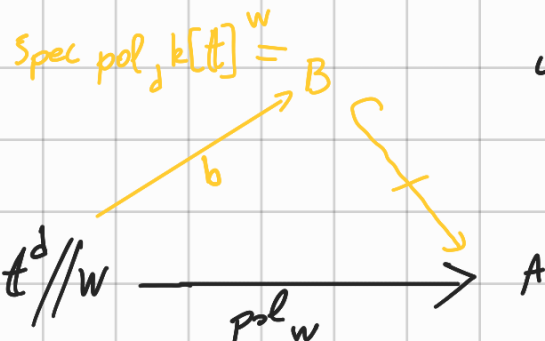
restriction $h := \text{pol}|_{\mathbb{C}_G^d} : \mathbb{C}_G^d \rightarrow A$

Similarly, $\mathfrak{t} \rightarrow \mathfrak{t} // W$ is W -inv (for $W \curvearrowright_{\text{adj}} \mathfrak{t}$) and G_m -equiv.

$\rightsquigarrow \text{pol}_W : \mathfrak{t}^d / W \rightarrow A$ for $W \curvearrowright_{\text{diag}} \mathfrak{t}^d$

Th 1: [Losik - Michor - Popov '06]: $k = \bar{k}$, $\text{char } k = 0$

Pol_W is finite and induces an injective map on k -points
 In other words, $\exists!$ reduced closed subscheme $B \subset A$ s.t.



where b is a universal homeo + normalisation.
 For $G = GL_m$, pol_W is a closed embedding
 and b an iso. $k[\mathfrak{t}^d]^W = \text{int. dos. of}$

Th 2: let $B \subset A$ as in Th 1

$\text{pol}_d k[t]^w$ in $k(t^d)^w$

$\exists B' \subset A$ closed subscheme, s.t. $B' \subset B$ is a thickening

s.t. $h: \mathbb{S}_G^d \rightarrow A$ factors through a map

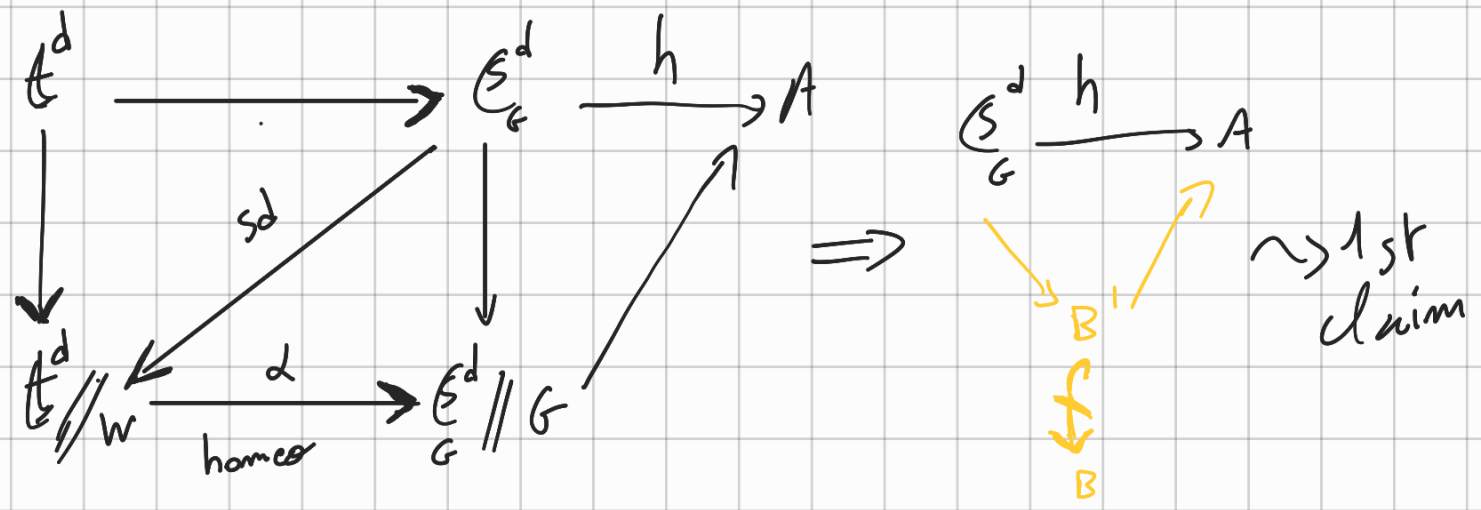
Sd' : $\mathbb{S}_G^d \rightarrow B'$

In particular, $\exists G(k)$ -equiv. morph $\mathbb{S}_G^d(k) \rightarrow t^d/w(k)$.

For $G = GL_n$, we have $B' = B$ and $Sd' = Sd$ (constructed in Deligne's theorem)

"proof:" (Chevalley restriction map is a homeo)

$t^d/w \xrightarrow[\text{homeo}]{} \mathbb{S}_G^d/G$ (Hunziker '97)



2nd claim follows from Deligne's theorem.

we've seen before that a Higgs bundle on a smooth proper alg. var.

over k can be represented by a map

III - spectral data morphism and Hitchin map

$$\begin{array}{ccc}
 \mathbb{T}_x^* & \begin{array}{c} [\mathbb{C}^d / (GL_d \times G)] \\ \downarrow \\ [A/GL_d] \end{array} & \xrightarrow{[h]} [A/GL_d] \\
 \downarrow & \nearrow \theta & \\
 X & \longrightarrow [BGL_d] &
 \end{array}$$

\leadsto Hitchin morphism: $h_x: \mathcal{M}_x \longrightarrow \mathcal{A}_x$

where $\mathcal{M}_x := \text{Mod. Space of Higgs bundles on } X$

$$\mathcal{A}_x := \left\{ \begin{array}{l} X \longrightarrow [A/GL_d] \text{ lying over} \\ \text{space } X \longrightarrow [BGL_d] \end{array} \right\} \cong \bigoplus_{i=1}^n H^0(X, S^{e_i} \Omega_x^1)$$

by choosing a system of homogeneous coordinates of degrees e_i

$$\text{def } \mathcal{B}_x := \left\{ \begin{array}{l} \text{closed} \\ X \longrightarrow [B/GL_d], B \text{ CA defined in Th 1 of} \\ \text{space} \\ \text{Lying over } X \longrightarrow [BGL_d] \end{array} \right\} \text{ Losik - Michor - papov}$$

called "Postulated image of the Hitchin map h_x ".

Actually \mathcal{B}_x is a closed subscheme of \mathcal{A}_x

Take $B' \hookrightarrow B$ the thickening from Th 2.

\leadsto a $\mathcal{B}'_x \hookrightarrow \mathcal{B}_x$ a thickening and of course

$$|\mathcal{B}'_x| = |\mathcal{B}_x|. \text{ (same top. sp.)}$$

Prop: Let $k = \bar{k}$, $\text{char } k = 0$,
 X d -dim smooth proper alg. var / k .

Then: \exists factorisation $h_x: \mathcal{M}_x \longrightarrow \mathcal{A}_x$

sd'_x is called "the spectral data morphism".



In part., $\forall \theta \in \mathcal{M}_x(k)$, $h_x(\theta) \in \mathcal{B}_x(k)$.

Proof: $\forall S \in \text{Sch}/k$, $\forall \theta \in \mathcal{M}_x(S)$

$$\theta: S \times X \longrightarrow [E^d / G \times GL_d]$$

$$h_x(\theta): S \times X \longrightarrow [A / GL_d]$$



\leadsto 1st claim.

Let $\theta \in \mathcal{M}(k)$.

X reduced, so $b': X \longrightarrow [B'/GL_d]$

because a smooth scheme over a field is regular, so locally a UFD,
 so in particular a domain, so it has no nilpotents.

$$\leadsto h_x(\theta) \in \mathcal{B}_x(k).$$



one of the conjectures of Chen-Ngô is that s_d' is surj.

Conjecture 2: [Chen-Ngô '20]

$$\forall b \in \mathcal{B}_x(k), h_x^{-1}(b) \neq \emptyset.$$

specular: True: for $d=2, G = GL_m,$

and

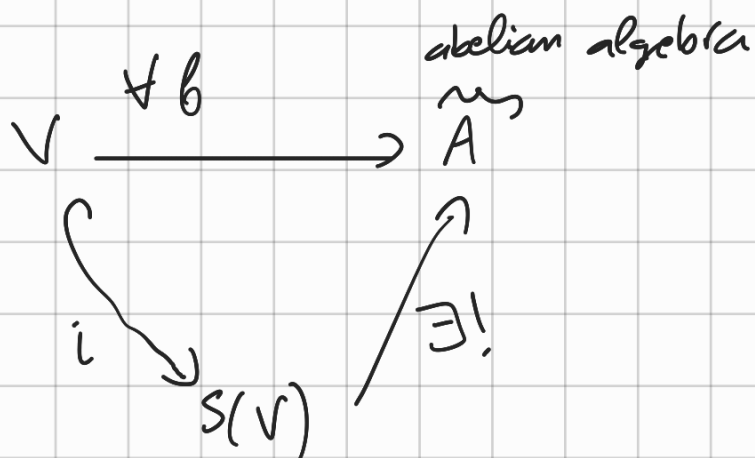
$$b \in \mathcal{B}^\nabla(k) := \left\{ \begin{array}{l} b: X \rightarrow [\mathcal{B}/GL_d] \text{ whose image} \\ \text{has non-empty intersection with} \\ [\mathcal{B}^\circ/GL_d] \end{array} \right\}.$$

$\mathcal{B}^\circ :=$ open dense locus of \mathcal{B} where $\mathbb{A}^d \rightarrow \mathcal{B}$
is a finite étale Galois with Galois group W .

Help: 1) a Cartan subalgebra is a maximal abelian subalgebra

$\mathfrak{h} \subset \mathfrak{g}$ s.t. $\forall \mathfrak{k} \in \mathfrak{h}$, $\text{ad}_{\mathfrak{k}}$ is semisimple

2) $S(V)$: is a commutative algebra:



$$\begin{aligned}
 S(V) &= \underbrace{T(V)}_{ii} / \langle x \otimes y - y \otimes x \rangle = \bigoplus_{k=0}^{\infty} \underbrace{T^k V}_{ii} / \langle x \otimes y - y \otimes x \rangle \\
 &= \bigoplus_{k=0}^{\infty} \underbrace{T^k V}_{ii} \\
 &= \underbrace{V \otimes \dots \otimes V}_{k \text{-times}}
 \end{aligned}$$

2) For $\dim X = 1$, torsion free rk 1 sheaf \Rightarrow line bundle

for $\dim X > 1$, it's not true anymore.

3) in $\dim 1$, torsion-free sheaf of rank 1 \Rightarrow loc. free
in $\dim > 1$, this is not true anymore

