Talk 4, 13-Dec 2022

I) Universal spectral data morph.
II) Weyl's polarisation
III) spectral data morphism and Hitchin map

- Setup: - X a proper smooth alg. var. of dim d over k.
  (we're no more restricted to dim 1)
  - T_X : tangent sheaf of X, Ω^1_X : sheaf of 1-forms on X
  - G: split reductive grp over k of rank n, $Lie(G) = \mathfrak{g}$

I) Universal spectral data morph.

Def: A G-Higgs bundle over X is a pair (E, θ) s.t.

1) $E \longrightarrow X$ is a G-bundle.
2) $\theta \in H^0(X, \operatorname{ad} E \otimes \Omega^1_X)$ seen as an $\mathcal{O}_X$-lin map $T_X \xrightarrow{\theta} \operatorname{ad}(E)$

$\left(\operatorname{ad} E \otimes_{\mathcal{O}_X} \Omega^1_X \cong \operatorname{ad} E \otimes T_X^* \cong \operatorname{Hom}_{\mathcal{O}_X}(T_X, \operatorname{ad} E)\right)$

s.t. $\forall\, v_1, v_2$ sections of T_X $[\theta(v_1), \theta(v_2)] = 0$. (integrability condition)

Def: the commuting scheme $\mathfrak{C}^d_G \subset \mathfrak{g}^d$ is the scheme theoretic zero fibre of the commutator map

$$\mathfrak{g}^d \longrightarrow \prod_{i<j} \mathfrak{g}$$
$$(\theta_1, \dots, \theta_d) \longmapsto \prod_{i<j} [\theta_i, \theta_j]$$

$\underbrace{\mathcal{E}^d_G(k)}_{\text{The } k\text{-points}} = \{ (\theta_1, \dots, \theta_d) \in \mathfrak{g}^d(k) \text{ s.t. } [\theta_i, \theta_j] = 0 , \quad 1 \le i,j \le d. \}$

note that the commuting relations are automatically satisfied for $d=1$.

we seek an alternative description of $\mathcal{E}^d_G$

k^d equipped with std basis $v_1, \dots, v_d$, $V_d := (k^d)^\vee$

$$\mathfrak{g}^d \xrightarrow{\sim} (V_d)^\vee \otimes \mathfrak{g}$$

$(\theta_1, \dots, \theta_d) \mapsto \theta : V_d \to \mathfrak{g}$ k-linear s.t. $\theta(v_i) = \theta_i$ (θ is unique)

$\rightsquigarrow \mathcal{E}^d_G :=$ closed subscheme of $\mathfrak{g}^d$ consisting of k-lin. maps

$\theta : V_d \to \mathfrak{g}$ s.t. $[\theta(v), \theta(v')] = 0 \quad \forall\, v, v' \in V_d$

$\rightsquigarrow GL_d \times G \circlearrowright \mathcal{E}^d_G$ (coming from natural $GL_d \circlearrowright V_d$, adjoint $G \circlearrowright \mathfrak{g}$)

$\xrightarrow{\text{quotient stack}} [\mathcal{E}^d_G / (GL_d \times G)] := \mathcal{H}$ called "Higgs stack"

it sends a test scheme S to the groupoid of triples

$$\mathcal{H} : S \longmapsto \left\{ (\mathcal{V}, \mathcal{E}, \theta) \;\middle|\; \begin{array}{l} \mathcal{V} \text{ is a rk-}d \text{ V.B. over } X \times S \\ \mathcal{E} : \text{a principal } G\text{-bundle over } X \times S \\ \theta : \mathcal{V} \to ad(\mathcal{E}) \text{ s.t. } [\theta(v), \theta(v')] = 0 \\ (\mathcal{O}_S\text{-linear}) \\ \forall\, v, v' \text{ local sections of } \mathcal{V} \end{array} \right\}$$

$S \in (Sch/k)^{op}$

using technical lemma from last talk, A Higgs field can be represented by θ lying over $X \to [BGL_d]$

$$\begin{tikzcd}
\text{cot. bundle } \{ T_X^* \arrow[d] & & {[\mathfrak{E}_G^d/(GL_d \times G)]} \arrow[d] \\
X \arrow[rru, dashed, "\theta"] \arrow[rr] & & {[BGL_d]}
\end{tikzcd}$$

$T_X^* \rightsquigarrow GL_d$-torsor

Recall: In dim 1 (last talk), we used Chevalley Kostant restriction theorem to derive a morphism of quotient stacks.

$$[\mathfrak{g}/G\times\mathbb{G}_m] \xrightarrow{[\chi]} [\mathrm{car}/\mathbb{G}_m] \quad \text{for } G \overset{\text{adj}}{\curvearrowright} \mathfrak{g},\ \mathbb{G}_m \overset{\text{homoth.}}{\curvearrowright} \mathfrak{g},\ \mathrm{car} = \mathfrak{g}/\!/G$$

to construct the Hitchin morphism in the language of stacks)

$$\rightsquigarrow h_X : \mathcal{M}_X \longrightarrow \mathbb{A}_X := H^0(X, \mathrm{car} \times^{\mathbb{G}_m} L_D)$$

$$\underbrace{\mathcal{M}_X}_{\substack{\text{called } H \\ \text{in last talk} \\ \text{(unfortunate)}}} : (Sch/k)^{op} \longrightarrow \mathrm{cat}$$

$$S \longmapsto \left\{ \begin{array}{l} h_a : X \times S \longrightarrow [\mathfrak{g}/G\times\mathbb{G}_m] \\ \text{above } h_D : X \times S \longrightarrow [B\mathbb{G}_m] \end{array} \right\}$$

In higher dim, it's similar, construction of the Hitchin map derives from G-inv. functions on $\mathfrak{E}_G^d$ as we will see with Weyl's polarisation construction, so studying the Hitchin map amounts to studying

$$\mathfrak{E}_G^d /\!/ G = \mathrm{Spec}\left(k[\mathfrak{E}_G^d]^G\right) \text{ GIT quotient, } G \overset{\text{diag}}{\underset{\text{adj}}{\curvearrowright}} \mathfrak{E}_G^d$$

$\boxed{[\mathcal{E}^d_G/G]}$ = quotient stack for $G \overset{\text{diag}}{\circlearrowright} \mathcal{E}^d_G$

$$[\mathcal{E}^d_G/G] \xrightarrow[q]{} \mathcal{E}^d_G/\!/G.$$

Now
in higher dim

$$\mathcal{M}_X : (Sch/k)^{op} \longrightarrow Cat$$

$$S \longmapsto \left\{ \begin{array}{l} \theta : X\times S \longrightarrow [\mathcal{E}^d_G/(G\times GL_d)] \\ \text{above } X\times S \longrightarrow [BGL_d] \end{array} \right\}$$

let $\mathfrak{t} \subset \mathfrak{g}$ be a cartan subalgebra $\left(\begin{array}{l}\text{in particular} \\ \mathfrak{t} \text{ is abelian}\end{array}\right)$

$\rightsquigarrow$ a factorisation

$$\begin{tikzcd} & \mathcal{E}^d_G \arrow[dr] & \\ \mathfrak{t}^d \arrow[ur] \arrow[rr] & & \mathfrak{g}^d \end{tikzcd}$$

embedding

$$\text{orbits}(W \overset{\text{diag}}{\circlearrowright} \mathfrak{t}^d) \subset \text{orbits}(G \overset{\text{diag}}{\circlearrowright} \mathcal{E}^d_G)$$

$$\rightsquigarrow \forall f \in k[\mathcal{E}^d_G]^G,\ f|_{\mathfrak{t}^d} \in k[\mathfrak{t}^d]^W$$

$$\rightsquigarrow \quad \mathfrak{t}^d/\!/W \xrightarrow{\alpha} \mathcal{E}^d_G/\!/G$$ (corresponding morph of aff. schemes)

Hunziker '97 : α is a universal homeomorphism

(i.e., finite morphism inducing a bijection on k points)

In particular, $\mathfrak{t}^d/\!/W$ is the normalisation of $(\mathcal{E}^d_G/\!/G)^{red}$

(the underlying reduced subscheme)

conjecture 1 [Chen – Ngô '20] : α is an iso.

(equivalently, $\mathcal{E}^d_G /\!/ G$ is reduced and normal).

Rem:

1) conjecture 1 $\iff$ $\mathcal{E}^d_G /\!/ G$ is reduced + normal. Indeed

$\overset{\text{easy}}{\underset{\text{part}}{\Longrightarrow}}$) if α is an iso, then

$$\mathfrak{t}^d /\!/ W \text{ reduced + normal} \Longrightarrow \mathcal{E}^d_G \text{ reduced + normal}$$

$\Longleftarrow$) if $\mathcal{E}^d_G /\!/ G$ reduced + normal, then α is a normalisation (by Hunziker '97) $\Longrightarrow$ α an iso

2) there is a long-standing conjecture saying that the scheme $\mathcal{E}^2_G$ is reduced. For $d \geq 3$ it seems to be doubtful.

But in general $\mathcal{E}^d /\!/ G$ behaves better as we will see.

Def: we call a **universal spectral data morphism** a G-inv map sd s.t. (making the following diagram commute)

$$\begin{tikzcd}
\mathfrak{t}^d \arrow[rr] \arrow[d] & & \mathcal{E}^d_G \arrow[d] \arrow[dll, "sd"'] \\
\mathfrak{t}^d /\!/ W \arrow[rr, "\alpha"'] \arrow[urr, phantom, "T1"] & & \mathcal{E}^d_G /\!/ G
\end{tikzcd}$$

(†) note that sd is completely indep. of X which is amazing

Rem: 1) $\exists$ of s_d is always satisfied by conjecture 1 + the morph

$$[\mathfrak{C}^d_G / G] \xrightarrow[q]{} \mathfrak{C}^d_G /\!/ G$$

since we don't have a proof of conjecture 3.1. and that this conjecture is crucial for the study of the Hitchin map, we state a weaker conjecture.

Conjecture 2 [Chen-NGÔ '20]: $\exists$ a G-inv map s_d s.t. $(\dagger)$ commutes.

Rem: conjecture 2 $\Rightarrow$ $\mathfrak{C}^d_G /\!/ G$ is reduced.

Indeed, the right triangle of $(\dagger)$ gives a commutative triangle of rings:

$$T_1 \rightsquigarrow \begin{tikzcd} & & k[\mathfrak{C}^d_G] \\ k[\mathfrak{t}^d]^W \arrow[urr, bend left] & & k[\mathfrak{C}^d_G]^G \arrow[ll] \arrow[u, hook, "\text{inclusion}"'] \end{tikzcd}$$

$\rightsquigarrow$ $k[\mathfrak{C}^d_G]^G \longrightarrow k[\mathfrak{t}^d]^W$ injective

$k[\mathfrak{t}^d]^W$ integral domain $\rightsquigarrow$ $k[\mathfrak{C}^d_G]^G$ integral dom.

(in particular reduced) $\blacksquare$

Th [Deligne] conjecture 2 holds for $G = GL_n$.

In particular, $\mathfrak{C}^d_G /\!/ G$ is reduced.

proof idea: construct S_d using the well-known fact that $k[\mathfrak{g}^d]^G$ is generated by $Tr(X(i_1)\cdots X(i_k))$ where $k \in \mathbb{Z}_{\geq 0}$, $1 \leq i_1, \ldots, i_k \leq d$

II) Weyl's polarisation

Roughly speaking Weyl's polarisation is a way to construct G-inv. functions on the space $\mathfrak{g}^d$ of d-tuples in $\mathfrak{g}$.

Given $c \in k[\mathfrak{g}]^G$, $x_1, \ldots, x_d \in k$, and

$$\mathfrak{g}^d \xrightarrow{\psi} k$$

let $(\theta_1, \ldots, \theta_d) \mapsto c(\theta_1 + \cdots + \theta_d)$

Let $pol_d\, k[\mathfrak{g}]^G$ be the subalgebra of $k[\mathfrak{g}^d]^G$ generated by all the ψ's.

Q: when does polarisations of G-inv. functions of $\mathfrak{g}$ generate the alg. of G inv. funcs on $\mathfrak{g}^d$: $pol_d[V]^G = k[V^d]^G$ hold? and this is a classical problem of invariant theory when you replace $\mathfrak{g}$ by V a general fin. dim. alg. G-module.

A: It depends, sometimes

- It does for instance for $G = \underbrace{O_m \circlearrowright V = k^m}^{\text{mat. action}}$ (Th. of Study)

$G = \underbrace{\mathfrak{S}_m \circlearrowright V = k^m}_{\text{permute coords}}$ (Th. of Weyl)

• It doesn't in general, for instance,

$$G = SL_n \overset{\text{nat}}{\circlearrowright} V = k^n$$

$$\rightsquigarrow k[V]^G = k \rightsquigarrow \mathrm{pol}_m k[V]^G = k \neq k[V^m]^G$$

Nevertheless polarisations are close to generate $k[V]^G$ as we have this result:

[Hunziker '97, Cor. 2.16]: Let G be finite,

let V be a fin. dim. alg. G-module, $m \in \mathbb{N}$.

Then $k[V^m]^G$ is the integral closure of $\mathrm{pol}_m k[V]^G$ in $k(V^m)^G$.

Let's formalize Weyl polarisation construction for some aff. alg. var.

For an affine variety Y with $\mathbb{G}_m \circlearrowright Y$,

$$F: R \in \mathrm{Alg}_k \longmapsto \{\mathbb{G}_m\text{-equiv map } V_d \otimes_k R \longrightarrow Y\}$$

F is representable by $Y^{V_d}_{\mathbb{G}_m} \in \mathrm{AffSch}$

E.g.:

1) $Y := \mathbb{A}^1_k$, $\mathbb{G}_m \circlearrowright Y$ via: $t.x = t^e x$

Then (by univ. prop. of symm. alg.) $Y^{V_d}_{\mathbb{G}_m} = \mathrm{Spec}(S^e S(V_d))$ (e-th symmetric tensor)

2) $Y := \mathfrak{g}$ then $Y^{V_d}_{\mathbb{G}_m}$ can be identified with $\mathfrak{g}^d$

3) $Y := \mathfrak{C} = \mathfrak{g} /\!/ G$, $\quad \mathfrak{C} \underset{\text{chevalley restriction}}{\cong}$ m-dim aff. space with homog. coord. $c_1, \dots, c_m$ of degrees $e_1, \dots, e_m$,

So $\mathfrak{C}^{V_d}_{\mathbb{G}_m} \cong \prod_{i=1}^{m} S^{e_i} \mathbb{A}^d =: A$ (this iso depends on the choice of the homog. coordinates)

Since $\mathfrak{g} \to \mathfrak{g}/\!/G$ is G-invariant and $\mathbb{G}_m$-equiv., (for $G \underset{\text{adj}}{\circlearrowright} \mathfrak{g}$)

$\overset{\text{induces}}{\rightsquigarrow}$ $pol : \mathfrak{g}^d \longrightarrow A$ (G-invariant for $G \overset{\text{diag}}{\circlearrowright} \mathfrak{g}^d$)

"Weyl's polarisation construction"

$\overset{\text{restriction}}{\rightsquigarrow}$ $h := pol|_{\mathfrak{C}^d_G} : \mathfrak{C}^d_G \longrightarrow A$

Similarly, $\mathfrak{t} \to \mathfrak{t}/\!/W$ is W-inv (for $W \underset{\text{adj}}{\circlearrowright} \mathfrak{t}$) and $\mathbb{G}_m$-equiv.

$\rightsquigarrow$ $pol_W : \mathfrak{t}^d / W \longrightarrow A$ for $W \overset{\text{diag.}}{\circlearrowright} \mathfrak{t}^d$

Th1: [Losik – Michor – Popov '06]: $k = \bar{k}$, char $k = 0$

pol_W is finite and induces an injective map on k-points

In other words, $\exists!$ reduced closed subscheme $B \subset A$ s.t.

$\operatorname{Spec} pol_d\, k[\mathfrak{t}]^W = B$

$$\begin{tikzcd} & B \arrow[dr, hook] & \\ \mathfrak{t}^d/\!/W \arrow[ur, "b"] \arrow[rr, "pol_W"'] & & A \end{tikzcd}$$

where b is a universal homeo + normalisation.

For $G = GL_m$, pol_W is a closed embedding and b an iso.

$k[\mathfrak{t}^d]^W$ = int clos. of

$p^d_d\, k[\mathfrak{t}]^W$ in $k(\mathfrak{t}^d)^W$

Th 2: Let $B \subset A$ as in Th 1

$\exists\, B' \subset A$ closed subscheme, s.t. $B' \subset B$ is a thickening

s.t. $h : \mathcal{E}^d_G \longrightarrow A$ factors through a map

$$sd' : \mathcal{E}^d_G \longrightarrow B'.$$

In particular, $\exists\, G(k)$-equiv. morph $\mathcal{E}^d_G(k) \longrightarrow \mathfrak{t}^d /\!/ W(k)$.

For $G = GL_m$, we have $B' = B$ and $sd' = sd$ (constructed in Deligne's theorem)

"proof": (Chevalley restriction map is a homeo)

$$\mathfrak{t}^d /\!/ W \underset{\text{homeo}}{\simeq} \mathcal{E}^d_G /\!/ G \qquad \text{(Hunziker '97)}$$

$$\begin{tikzcd}
\mathfrak{t}^d \arrow[r] \arrow[d] & \mathcal{E}^d_G \arrow[r, "h"] \arrow[d] \arrow[dl, "sd"'] & A \\
\mathfrak{t}^d /\!/ W \arrow[r, "\alpha", "\text{homeo}"'] & \mathcal{E}^d_G /\!/ G \arrow[ur] &
\end{tikzcd}
\implies
\begin{tikzcd}
\mathcal{E}^d_G \arrow[r, "h"] \arrow[dr] & A \\
& B' \arrow[u] \arrow[d, "f"] \\
& B
\end{tikzcd}
\rightsquigarrow \text{1st claim}$$

2^{nd} claim follows from Deligne's theorem.

we've seen before that a Higgs bdle on a smooth proper alg. var. over k can be represented by a map

III - spectral data morphism and Hitchin map

$$\begin{tikzcd} T_X^* \arrow[d] & {[\mathfrak{C}_G^d/(GL_d \times G)]} \arrow[d] \arrow[r, "{[h]}"] & {[A/GL_d]} \\ X \arrow[ur, dashed, "\theta"] \arrow[r] & {[BGL_d]} & \end{tikzcd}$$

$\rightsquigarrow$ Hitchin morphism : $h_X : \mathcal{M}_X \longrightarrow \mathcal{A}_X$

where $\mathcal{M}_X :=$ Mod. Space of Higgs bdles on X

$$\mathcal{A}_X := \left\{ \begin{array}{l} X \longrightarrow [A/GL_d] \text{ lying over} \\ X \longrightarrow [BGL_d] \end{array} \right\} \simeq \bigoplus_{i=1}^{n} H^0(X, S^{e_i}\Omega_X^1)$$

space

by choosing a system of homogeneous coordinates of degrees e_i

def $\mathcal{B}_X := \left\{ \begin{array}{l} X \longrightarrow [B/GL_d], B \subset A \text{ closed defined in Th 1 of Losik - Michor - popov} \\ \text{lying over } X \longrightarrow [BGL_d] \end{array} \right\}$

space

called "Postulated image of the Hitchin map h_X".

Actually $\mathcal{B}_X$ is a closed subscheme of $\mathcal{A}_X$

Take $B' \hookrightarrow B$ the thickening from Th 2.

$\rightsquigarrow$ a $\mathcal{B}'_X \hookrightarrow \mathcal{B}_X$ a thickening and of course

$$|\mathcal{B}'_X| = |\mathcal{B}_X| \quad (\text{same top. sp.})$$

Prop: let $k = \bar{k}$, char $k = 0$,

X d-dim smooth proper alg. var $/ k$.

Then: $\exists$ factorisation

$$\begin{tikzcd} \mathcal{M}_X \arrow[rr, "h_X"] \arrow[dr, "sd'_X"'] & & \mathcal{A}_X \\ & \mathcal{B}'_X \arrow[ur] & \end{tikzcd}$$

sd'_X is called "the spectral data morphism".

In part., $\forall \theta \in \mathcal{M}_X(k)$, $h_X(\theta) \in \mathcal{B}_X(k)$.

Proof: $\forall S \in Sch/_k$, $\forall \theta \in \mathcal{M}_X(S)$

$$\theta : S \times X \longrightarrow [\mathcal{E}^d_G / G \times GL_d]$$

$$\begin{tikzcd} S \times X \arrow[rr, "h_X(\theta)"] \arrow[dr, dashed, "b'"'] & & {[A/GL_d]} \\ & {[B'/GL_d]} \arrow[ur, dashed] & \end{tikzcd}$$

(by Th. 2)

$\rightsquigarrow$ 1st claim.

let $\theta \in \mathcal{M}(k)$.

$$\begin{tikzcd} & {[B/GL_d]} \arrow[d, bend left] \\ X \arrow[ur, dashed, "b"] \arrow[r, "b'"'] & {[B'/GL_d]} \end{tikzcd}$$

X reduced, so $b' : X \to [B'/GL_d]$

because a smooth scheme over a field is regular, so locally a UFD, so in particular a domain, so it has no nilpotents.

$\rightsquigarrow \quad h_X(\theta) \in \mathcal{B}_X(k).$ $\blacksquare$

one of the conjectures of Chen-Ngô is that sd' is surj.

<u>Conjecture 2:</u> [Chen-Ngô-'20]

$$\forall\, b \in \mathcal{B}_X(k), \quad h_X^{-1}(b) \neq \phi.$$

spoiler: <u>True:</u> for $d=2$, $G = GLm$,

and

$$b \in \mathcal{B}^{\heartsuit}(k) := \left\{ \begin{array}{l} b : X \to [B/GL_d] \text{ whose image} \\ \text{has non-empty intersection with} \\ [B^{\circ}/GL_d] \end{array} \right\}.$$

B° := open dense locus of B where $\mathbb{A}^d \longrightarrow B$

is a finite étale Galois with Galois group W.

Help: 1) a cartan subalg is a maximal abelian subalgebra

$\mathfrak{t} \subset \mathfrak{g}$ s.t. $\forall t \in \mathfrak{t}$, ad_t is semisimple

2) $S(V)$: is a commutative algebra :

abelian algebra

$$\begin{tikzcd}
V \arrow[rr, "\forall f"] \arrow[dr, bend right, "i"'] & & A \\
& S(V) \arrow[ur, "\exists!"'] &
\end{tikzcd}$$

$$S(V) = \underbrace{T(V)}_{\underbrace{\bigoplus_{k=0}^{\infty} \underbrace{T^k V}_{\underbrace{V \otimes \dots \otimes V}_{k\text{-times}}}}} \Big/ \langle x \otimes y - y \otimes x \rangle = \bigoplus_{k=0}^{\infty} T^k V \Big/ x \otimes y - y \otimes x$$

2) for $\dim X = 1$, torsion free rk 1 sheaf $\Rightarrow$ line bdle

for $\dim X > 1$, it's not true anymore.

3) in dim 1, torsion-free sheaf of rank 1 $\Rightarrow$ loc. free

in dim > 1, this is not true anymore