Talk Jorge

Prop: $\forall b \in B_X^{\heartsuit}(k)$, the fiber $h_X^{-1}(b)$ of the Hitchin map is $\simeq$ to a stack of max Cohen-Macaulay sheaves of generic rank m on the spectral curve X_b.

I) $G = GL_2$, $d \geq 2$.

B subscheme of $A = \mathbb{A}^2 \times S\mathbb{A}^2 \times \dots \times S_m \mathbb{A}^m$

$B \simeq \mathrm{Chow}(\mathbb{A}^2)$

$b : X \to \mathrm{Chow}(T_X^* \setminus X)$

$\mathrm{Chow}(T_X^*), \dots, Q(T_X^* \setminus X)$ as its complement.

A point of $\mathrm{Hilbert}(\mathbb{A}^2)$ is a 0-dim subscheme Z of $\mathbb{A}^2$ s.t.

$$Z \simeq \bigsqcup_{a \in \mathbb{A}^2} Z_a$$

where Z_a is a 0 dim subscheme of $\mathbb{A}^2$ whose closed pt is a.

$$HC_m : \mathrm{Hilb}(\mathbb{A}^2) \longrightarrow \mathrm{Chow}(\mathbb{A}^2)$$

$$Z \longrightarrow \sum \mathrm{length}(Z_a) \cdot a.$$

$HC_{T^*_X} : \mathrm{Hilb}(T^*_X \setminus X) \longrightarrow \mathrm{Chow}(T^*_X \setminus X)$ $(\star)$

$(\star)$ is proper $(*)$ is isomorphic over $\mathrm{Chow}^\circ(T^*_X \setminus X)$.

Let X be smooth surface / k.

// Z // closed subscheme of codim ≥ 2 of X.

// $j : U \hookrightarrow X$ the open immersion. The functor $V \longrightarrow j_* V$

is an equiv of cat.

$HC_{T^*_X} : \mathrm{Hilb}(T^*_X \setminus X) \longrightarrow \mathrm{Chow}(T^*_X \setminus X)$ ✱

✱ proper : isomorphic over $\mathrm{Chow}^\circ(T^*_X \setminus X)$

proof: X smooth surface over k

$Z \hookrightarrow X$ of codim ≥ 2

$U = X \setminus Z$

$U \hookrightarrow X$

the functor $V \longrightarrow j_* V$ is an equiv. of cats

of locally free sheaves on U and on X.

$\mathcal{F}$ loc. free sheaf on U.

$\mathcal{G}$ coh sheaf on X s.t. $\mathcal{G}|_U \simeq \mathcal{F}$

$\hat{\mathcal{F}} = \mathcal{G}^{\wedge\wedge}$, $j_*(j^* \hat{\mathcal{F}}) = \hat{\mathcal{F}}$ and $\hat{\mathcal{F}}|_U = \mathcal{F}$

$j_* \mathcal{F}$ is reflexive sheaf $\Rightarrow j_* \mathcal{F}$ loc free. ∎

prop: $\forall\, b \in B^\heartsuit_X(k)$, $\exists!$ finite flat cov.

$p^{CM}_b : X_b \longrightarrow X$ of deg. n with a X-morph

$\iota : X_b^{CM} \longrightarrow T_x^*$ satisfying

- $\exists$ $U \hookrightarrow X$, $Z = X \setminus U$ of dim ≥ 2 s.t. U is a closed embedding on U
- $\iota : X_x^{CM} \to T_x^*$ factors through the closed subscheme $X_b^\circ \subset T_X^*$ and the morphism $q_b^{CM} : X_b \longrightarrow X_b^\circ$ is finite Cohen-Macaulayfication of X_b.

Denote U° = the inverse image of $\mathrm{Chow}^\circ(T_X^*)\ [b^{-1}(\mathrm{Chow}^\circ(T^*))]$

Using HC morphism we can lift to a unique morph.

$$b^\circ_{H_1} : U^\circ \longrightarrow \mathrm{Hilb}$$

U° whose complement is a closed subscheme of codim ≥ 2.

$$b^U : U \longrightarrow \mathrm{Hilb}(T_X^* \setminus X) \times U.$$

$\rightsquigarrow$ get a finite flat morph $U_b^+ \longrightarrow U$ of degree n with a closed embedding $\iota : U_b^+ \longrightarrow T_U^*$.

$\rightsquigarrow$ $\exists$ a unique finite flat map : $p_b^{CM} : X_b^{CM} \longrightarrow X$ of degree n, we can also get the morphism $\iota : X_b^{CM} \longrightarrow T_X^*$

smoothness $\rightsquigarrow$ of X

this map is a Cohen-Macaulayfication (no need to check the cohom. conds)

Using Cayley-Hamilton, the vector bundle $p_{b^*} \mathcal{O}_{X_b}^{CM}$ as $\mathcal{O}_{T^*_X}$-mod over T^*_X is supported by X_b.

$\rightsquigarrow X_b^{CM} \longrightarrow T^*_X$ factors through X_b°:

$$q_b^{CM} : X_b^{CM} \longrightarrow X_b^\circ$$

$U^\circ = b^{-1}(\mathrm{Chow}^\circ(T^*_X \setminus X))$, Z = his complement.

Let η be a generic pt of an irreducible component of Z of dim 1.

Let X_η = Localisation. By restriction $p_{b_\circ} \mathcal{O}_{X_b^\circ}$ to $\mathcal{O}_{X_\eta}$, we get finite flat module..

Consider the quotient $\mathrm{Spec}\left(p_{b_\circ} \mathcal{O}_{X_b^\circ} / \left(p_{b^*} \mathcal{O}_{X_{b^*}}^{tor}\right)\right)$ and thus a section:

$$\hat{b} : X_\eta \longrightarrow \mathrm{Hilb}(T^*_X \setminus X) \times X_\eta \text{ over } b|_{X_\eta} = b.$$

$$\mathrm{Spec}\left(p_{b^\circ} \mathcal{O}_{X_b^\circ} / (p_b \mathcal{O}_{X_b})^{tor}\right) \simeq \mathrm{Spec}\left(p_b^{CM} \mathcal{O}_{X_b^{CM}}\right).$$

$\boxtimes$

for every $b \in B_X^{\heartsuit}(k)$, the fiber $b_X^{-1}(b)$ is iso to the stack of cohen Macaulay sheaves $\mathcal{F}$ of generic rank 1 over the cohen Macaulay spectral surf X_b^{CM}.

Let $(E, \tau) \in \mathcal{M}_X$ be a Higgs bundle over X lying over $b \in B_X^{\heartsuit}(k)$.

$\theta : \mathcal{T}_X \longrightarrow End(E)$.

$$S(\mathcal{T}_X) \longrightarrow End(E) \xrightarrow[\text{Hamilton}]{\text{cayley}} \text{factors thr. } P_{b*} \mathcal{O}_{X_b}.$$

Let U°, Z, η be as before.

$$P_{b*} \mathcal{O}_{X_b} \otimes \mathcal{O}_{X_\eta} \longrightarrow End(E) \otimes \mathcal{O}_{X_\eta}$$

$\exists$ an open U whose complement is codim ≥ 2.

$$\rightsquigarrow \quad P_b^{CM} \mathcal{O}_{X_b^{CM}} \otimes \mathcal{O}_U \longrightarrow End(E) \otimes \mathcal{O}_U$$

Using Serre theorem $:\rightsquigarrow P_b^{CM} \mathcal{O}_{X_b^{CM}} \longrightarrow End(E)$.

$E = P_{b*} F$ where F is cohen-Macaulay $\mathcal{O}_{X_b^{CM}}$-mod of generic rank 1.

Rem: . $b^{-1}(k) \neq \phi$., In particular, $b^{-1}(k)$ contains the Picard stack $\mathcal{P}$ of line bundle on X_b^{CM}.

let $L \in Pic(X_b^{CM})$

$(*) \, p_b^{CM} \Rightarrow p_b^{CM\,*} L \Rightarrow h_X^{-1}(b) \ni \mathcal{P}$.

$L \in \mathcal{P}$, $F \in h_x^{-1}(b)$ coh-Macaulay.

$(L, F) \longrightarrow L \otimes_{X^{CM}} \mathcal{F}$.

Define: $B^{\diamond}(k) \subset B_X^{\heartsuit}(k)$ points s.t. the corresponding cohen-Macaulay spectral surf is normal.

$*$ For $b \in B_X^{\diamond}(k)$, the action of $\mathcal{P}$ on $h_X^{-1}(b)$ is free.