Let X a smooth surface; C a smooth proper curve.

$\pi : X \to C$ proper flat surj. map

X^o: largest open subset of X.

Such that the restriction of π to X^o is smooth.

consider $\quad d\pi : T_C^* \times_C X \longrightarrow T_X^*$

$\overset{\text{induce}}{\rightsquigarrow} \quad [d\pi] : \mathrm{Chow}_m(T_C^*/C) \times_C X \longrightarrow \mathrm{Chow}_m(T_X^*/X)$

For every section $b_C : C \longrightarrow \mathrm{Chow}_m(T_C^*/C)$

We construct $\quad X \cong C \times_C X \xrightarrow{b_C \times \mathrm{id}_X} \mathrm{Chow}_m(T_C^*/C) \times_C X$

$$\xrightarrow{[d\pi]} \mathrm{Chow}_m(T_X^* X)$$

which is a section of $\mathrm{Chow}_m(T_X^*/X) \longrightarrow X$.

The assignment $b_C \mapsto b_X$ defines a map

$$\begin{array}{ccc} \mathcal{B}_C & \xrightarrow{\iota_\pi} & \mathcal{B}_X \\ \downarrow & & \downarrow \\ \mathcal{A}_C & \xrightarrow{j_\pi} & \mathcal{A}_X \cong \bigoplus H^0(X, S^i \Omega^1_X) \end{array}$$

$\oplus H^0(X, \Omega^1_C)$

$\rightsquigarrow \mathcal{B}_C \subseteq \mathcal{B}_X$ as a subspace

Since $d\pi$ is a closed embedding over X°, we get

$${}^\circ\mathcal{B}_X^\heartsuit = \mathcal{B}_C \cap \mathcal{B}_X^\heartsuit$$

<u>Recall:</u> $C_b^\bullet \longrightarrow C$ the spectral curve, define

$X_b^+ := C_b^\bullet \times_C X$ and $p_b^+ : X_b^+ \longrightarrow X$

<u>Lemma:</u> $\exists$ a finite X-morphism $X_b^+ \longrightarrow X_b^\bullet$ and if $b \in \mathcal{B}_C$ then this is an iso. If we assume further that π has only reduced fibers, then

$$q_b^+ \cong q_b^{CM} \text{ for } b \in \mathcal{B}_C^\heartsuit$$

Proof:

$$\begin{array}{ccc} X_b^+ & \xrightarrow{\;i_b^+ := d_\pi|_{X_b^+}\;} & T^*_X \\ & {\scriptstyle q_b^+}\searrow \quad \nearrow & \\ & X_b^\circ & \end{array}$$

if we assume all fibers of π are reduced, then

$X \setminus X^\circ$ has codim 2, then, then use 7.2 from last talk.

Define: $\mathcal{B}_C^\diamond$ to be the open subset of points $b \in \mathcal{B}_C^\heartsuit$ for which C_b is smooth + irred.,

Cor: Again under the assumption that π only has red. fibers, we have the inclusion $\mathcal{B}_C^\diamond \subset \mathcal{B}_X^\diamond$.

X_b^{CM} is normal $\forall\, b \in \mathcal{B}_C^\diamond$

proof: Since X_b^{CM} is CM, we only need to show that X_b^+ is smooth in codim ≤ 1.

Since π has only reduced fibers, $X \setminus X^\circ$ has codim 2.

Define further $X_b^{+\circ} = \tilde{Z}_a \times_C X^\circ \subseteq X_b^+$ which is smooth.

$X_b^+ \setminus X_b^{+\circ}$

<u>prop:</u> Let X smooth proj and C is either ruled or elliptic. non-isotrivial & If π has only reduced fibers, then

$\forall\, m$, $\exists$ an iso $H^0(C, S^m \Omega_C^1) \longrightarrow H^0(X, S^m \Omega_X^1)$

<u>Proof:</u> $\pi : X \to C$ is proper flat.

X smooth proj and the gen. fiber is smooth of genus 1.

with π non iso-trivial and relatively minimal.

<u>Recall:</u> X° the open locus $\Rightarrow X \setminus X^\circ$ is a dim 0

subscheme and over X° we get

[1] $0 \to \mathcal{T}_{X^\circ/C} \longrightarrow \mathcal{T}_{X^\circ} \longrightarrow (\pi|_{X^\circ})^* \mathcal{T}_C \longrightarrow 0$

And from this we get

$$0 \to S^{m-1}\mathcal{J}_{X^0} \otimes \mathcal{J}_{X^0/C} \longrightarrow S^m \mathcal{J}_{X^0} \to (\pi|_{X^0})^* S^m \mathcal{J}_C \to 0$$

now η a gen. pt. of C, we get $X_\eta = X \times_C \eta$

$\mathcal{J}_X|_{X_\eta}$ is obtained of the trivial bundle on X_η

by restriction [1] to X_η that is non trivial since π is

non-isotrivial

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$$\rightsquigarrow \quad 0 \to \mathcal{O}_{X_\eta} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{X_\eta} \to 0$$

we'll need two lemmas:

<u>Lemma 1</u> the exact seq obtained from [2]

$$0 \longrightarrow S^{m-1}\mathcal{E} \longrightarrow S^m\mathcal{E} \longrightarrow \mathcal{O}_{X_\eta} \to 0 \quad [2]$$

is not split.

<u>Proof:</u> suppose $0 \to \mathcal{L}' \to \mathcal{E} \to \mathcal{L} \to 0$ [3]

w/ $\mathcal{L}, \mathcal{L}'$ are line bundles

$\rightsquigarrow$ canonical filtration

$$0 = \mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_n = S^n \mathcal{E} \quad \text{s.t.}$$

$$\mathcal{F}_i \cong S^i \mathcal{E} \otimes \mathcal{L}'^{\otimes n-i} \quad \text{and} \quad \mathcal{F}_i / \mathcal{F}_{i-1} \cong \mathcal{L}^{\otimes i} \otimes \mathcal{L}'^{\otimes n-i}$$

$\otimes$ [3] with $\mathcal{L}^{\otimes n-1}$ $\rightsquigarrow$

$$0 \to \mathcal{F}_{n-1} / \mathcal{F}_{n-2} \longrightarrow \mathcal{F}_n / \mathcal{F}_{n-2} \longrightarrow \mathcal{F}_n / \mathcal{F}_{n-1} \longrightarrow 0$$

if [3] don't split then $0 \to \mathcal{F}_{n-1} \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{L}^{\otimes n}$

Taking $\mathcal{L}' = \mathcal{L} = \mathcal{O}_{X_\eta}$, result follows ☐

Lemma 2

$\forall n \in \mathbb{N}$, we have:

- $\dim \operatorname{Ext}^1(\mathcal{O}_{X_\eta}, S^n \mathcal{E}) = 1$
- $\dim \operatorname{Hom}(S^n \mathcal{E}, \mathcal{O}_{X_\eta}) = 1$
- restriction $\operatorname{Hom}(S^n \mathcal{E}, \mathcal{O}_{X_\eta}) \longrightarrow \operatorname{Hom}(S^{n-1} \mathcal{E}, \mathcal{O}_{X_\eta})$ is 0.

proof: $\operatorname{Ext}^{\bullet}(-, \mathcal{O}_{X_\eta})$ give LES from where we can read of the statement by induction. ☐

continuation of prop proof :

$$H^0(C, S^m\Omega^1_C) \longrightarrow H^0(X, S^m\Omega^1_X).$$

to show surjectivity, take $\alpha \in H^0(X, S^m\Omega^1_X)$

$\overset{\alpha}{\rightsquigarrow}$ $S^m \mathcal{T}_X \to \mathcal{O}_X$ and after restricting to η:

$$\alpha_\eta : S^m E \longrightarrow \mathcal{O}_{X_\eta}$$

$$0 \to S^{m-1}\mathcal{T}_{X^0} \otimes \mathcal{T}_{X^0/C} \to S^m \mathcal{T}_{X^0} \to (\pi|_{X^0})^* S^m \mathcal{T}_C \to 0$$

The restriction factors through $(\pi|_{X^0})^* S^m \mathcal{T}_C$. But since $X \setminus X^0$ has dim. 0, this factorisation is actually goes through $\pi^* S^m \mathcal{T}_C$, that is α comes from a symmetric form on C so the map $H^0(C, S^m\Omega^1_C) \xrightarrow{\sim} H^0(C, S^m\Omega^1_X)$.

■

Some consequences:

- $\mathcal{A}_C = \mathcal{A}_X$ and $\mathcal{B}_C = \mathcal{A}_C$ we get $\mathcal{B}_C = \mathcal{B}_X$ with $\mathcal{B}_X^\diamond$ and $\mathcal{B}_X^\heartsuit$ are open dense subsets of $\mathcal{B}_X$.

Moreover, $X_b^\bullet$ is a finite scheme over X embedded in T_X^*

$$X_b^{CM} = X_b^{+} = C_b \times_C X$$

But for elliptic surfaces, $X_b^{CM} \to X_b^{\circ}$ may not be iso.

E.g. (8.1 from Ngô-chen paper)