Θ-stratification of Λ-Coh^d

Recall: R is a DVR with uniformizer ϖ

our notation $\quad Y_{\Theta_R} := \mathrm{Spec}(R[t]), \quad Y_{\overline{ST}_R} := \mathrm{Spec}\,\dfrac{R[s,t]}{(st-\varpi)}$

$$\Theta_R := [Y_{\Theta_R}/\mathbb{G}_m], \quad \overline{ST}_R := [Y_{\overline{ST}_R}/\mathbb{G}_m]$$

$\mathcal{M}$ a stack.

$$B\mathbb{G}_m^q := [\mathrm{Spec}\,\mathbb{Z}/\mathbb{G}_m^q]$$

k is a field.

§0. polynomial numerical invariant

choose $(L_m)_{m\in\mathbb{Z}}$, $L_m \in \mathrm{Pic}(\mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $\forall m \in \mathbb{Z}$.

$$\forall g : (B\mathbb{G}_m^q)_k \longrightarrow \mathcal{M}$$

$$g^*(L_m) \longrightarrow (B\mathbb{G}_m^q)_k = \left[\frac{\mathrm{Spec}\,\mathbb{Z}}{\mathbb{G}_m^q}\right] \longleftrightarrow \star$$

skip

$$\star \left\{ \begin{array}{l} \mathbb{G}_m^q\text{-torsor}: \; Y \longrightarrow g^*\mathcal{L}_m \\ \mathbb{G}_m^q\text{-eq. map}: \; Y \longrightarrow \mathrm{Spec}\,\mathbb{Z}. \end{array} \right.$$

$g^*(L_m) \rightsquigarrow c \in X^*(\mathbb{G}_m^q) \otimes \mathbb{Q} \cong \mathbb{Q}^q$

So we can interpret $g^*(\mathcal{L}_m)$ as a q-tuple of rational nbrs

$\rightsquigarrow c = (w_m^{(i)})_{i=1}^q$, "the weight of $g^*(\mathcal{L}_m)$".

Fix i $\rightsquigarrow$ $w_m^{(i)} = P(m) \in \mathbb{Q}[m]$

Turns out that in most cases we can choose the line bdls L_m s.t.

In which case, $\forall g : (B\mathbb{G}_m^q)_k \to \mathcal{M}$ $\overset{\text{define}}{\rightsquigarrow}$ $L_g : \mathbb{R}^q \to \mathbb{R}[m]$

$$\begin{pmatrix} r_1 \\ \vdots \\ r_q \end{pmatrix} \mapsto \sum_{i=1}^q r_i w_m^{(i)}$$

$\rightsquigarrow$ polynomial numerical invariant:

$\underbrace{(\text{the induced } (\mathbb{G}_m^q)_k \xrightarrow{\gamma} \mathrm{Aut}(g)|_{\mathrm{Spec}\, k} \text{ has fin. kernel})}$

$\forall g : B\mathbb{G}_m^q \to \mathcal{M}$ non-degenerate, $\quad \nu_\gamma(\vec{r}) = \dfrac{L_g(\vec{r})}{\sqrt{b_\gamma(\vec{r})}}$

where $b_\gamma(-)$ is a positive definite quadratic norm making ν scale-invariant.

Recall how we use numerical invariant to define the ss-locus

Recall: • $f \in \mathrm{Filt}(\mathcal{M})$ is non deg. if the restriction

$$f|_0 : [0/(\mathbb{G}_m)_k] \longrightarrow \mathcal{M} \text{ is non deg.}$$

• $p \in |\mathcal{M}|$ is semistable if $\forall f \in \mathrm{Filt}(\mathcal{M})$ non deg. with $f(1) = p$ $\nu(f) \leq 0$. O/W p is unstable

• stability function on $\mathcal{M}$

. $M^\nu(p) = \sup\{\mu(f) : f \in |\mathrm{Filt}(\mathcal{M})| \text{ s.t. } f(1) = p\} \in \Gamma \sqcup \{-\infty\}$,

For p stable

. $M^\nu(p) = -\infty$ for p unstable

• $\forall c \in \mathbb{R}[n]_{\geq 0}$, $\mathcal{M}_{\leq c} := \{p \in |\mathcal{M}| : M^\nu(p) \leq c\} \overset{\text{open}}{\subset} \mathcal{M}$

• Question: when does ν define a Θ-stratification on $\mathcal{M}$.?

§1 - S-monotonicity, Θ-monotonicity

Def: A polynomial numerical invariant ν on a stack $\mathcal{M}$ is strictly Θ-monotone (resp. strictly S-monotone) if:

Set $\mathcal{X} := \Theta_R$ (resp $\mathcal{X} := \overline{ST}_R$)

Let $\varphi : \mathcal{X} \setminus (0,0) \longrightarrow \mathcal{M}$

Then up to replacing R with a finite DVR extension,

Then $\exists\ f : \Sigma \longrightarrow Y_{\mathcal{X}}$ and $\tilde{\varphi} : [\Sigma / \mathbb{G}_m] \longrightarrow \mathcal{M}$, s.t.

with Σ a reduced and irreducible $\mathbb{G}_m$-equivariant scheme.

(M_1) : f is proper, $\mathbb{G}_m$-equivariant restricting to

$f : \Sigma_{Y_{\mathcal{X}} \setminus (0,0)} \xrightarrow{\cong} Y_{\mathcal{X}} \setminus (0,0)$ away from $(0,0)$

(M_2) :

$$\begin{tikzcd}
{[(\Sigma_{Y_{\mathcal{X}} \setminus (0,0)}) / \mathbb{G}_m]} \arrow[d, "f"'] \arrow[dr, "\tilde{\varphi}"] & \\
\mathcal{X} \setminus (0,0) \arrow[r, "\varphi"'] & \mathcal{M}
\end{tikzcd}$$

(M_3) : if k is a finite extension of the residue field of R.

$\forall a \geqslant 1$, $\forall\ \mathbb{P}^1_k[a] \longrightarrow \Sigma_{(0,0)}$ finite $\mathbb{G}_m$-equiv.

we have $\nu(\tilde{\varphi}|_{[\infty/\mathbb{G}_m]}) \geqslant \nu(\tilde{\varphi}|_{[0/\mathbb{G}_m]})$ $\Big|$ $0 := [0:1]$, $\infty := [1:0]$

($\mathbb{P}^1_k[a]$ is $\mathbb{P}^1_k$ equipped with $\mathbb{G}_m$-action : $t \cdot [x:y] = [t^{-a}x : y]$)

In classical GIT, the Hilbert-Mumford criterion is defined using a single line bundle. In ∞-dim GIT, we have seen earlier that we rather use an ∞ sequence of line bundles, and the construction served for the general case so let's define now a polynomial numerical invariant specifically tailored for the purpose of studying the stacks $\mathrm{Coh}^d(X)$ and its subfunct of T-pure sheaves of dim. d equipped with Λ-mod structure

Given the seq of line bundle $(L_m)_{m\in\mathbb{Z}}$ we can efine a polynomial numerical invariant

<u>Def</u>: Let $f: \Theta_k \longrightarrow \mathrm{Coh}^d(X)$ a non-degenerate filtration given by $(F_m)_{m\in\mathbb{Z}}$ (recall this comes from Rees construction)

$b(g)(v) := \sum_{m\in\mathbb{Z}^q} \mathrm{rk}_{F_m^{\Rightarrow}} \cdot (\underset{\mathrm{std}}{m \cdot} v)^2$ | $F_{m+1} \subset F_m$, F_m/F_{m+1} is d-pure, $F_m = 0$ for $m \gg 0$, $F_m = F$ for $m \ll 0$

Polynomial numerical invariant:

$$\nu(f) := \frac{wt(L_m|_0)}{\sqrt{b(f|_0)}} = \frac{\sum_{m\in\mathbb{Z}} m.\left(\bar{p}^{(m)}_{F_m/F_{m+1}} - \bar{p}^{(m)}_{F}\right).\mathrm{rk}_{F_m/F_{m+1}}}{\sqrt{\sum_{m\in\mathbb{Z}} \mathrm{rk}_{F_m/F_{m+1}} \cdot m^2}}$$

we use same formula for ν on $\Lambda\,\mathrm{coh}^d(X)$ by pullbacking ν along the forget functor

we prove the monotonicity of the polynomial numerical invariants using the techniques of ∞-dim GIT we studied before, namely the affine Grassmannians and the rational filling conditions introduced by Martin

<u>Theorem</u>: SPS $X \longrightarrow S$ is flat with geometrically integral fibers of dim. d. Then ν is strictly Θ-monotone and strictly S-monotone on $\mathcal{M} = \mathrm{Coh}^d(X)$ (or $\Lambda\,\mathrm{Coh}^d(X)$ or $\mathrm{Pair}^d_{\mathcal{A}}(X)$)

<u>Proof</u>:

Set $\mathcal{M} = \mathrm{Coh}^d(X)$, $\Lambda\,\mathrm{Coh}^d(X)$ or $\mathrm{Pair}^d_{\mathcal{A}}(X)$. (doesn't matter, the proof is same)

R is a complete DVR, $Y := Y_{\mathcal{O}_R}$ (resp. $Y_{\overline{ST}_R}$).

the starting point in the definition of monotonicity is to choose

As proved by Martin, $Coh^d(X)$, $\Lambda Coh^d(X)$ admit $\mathcal{O}_R$ and $\overline{ST}_R$ rational filling

Rational filling for $\mathcal{M}$

$$\begin{tikzcd}
Y \setminus (0,0) \arrow[dr, "\tau"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "j"] & & \\
& \mathcal{M} \times_{\mathcal{M}_{rat}} Y \arrow[r, "Forget"] \arrow[d] & \mathcal{M} \arrow[d] \\
& Y \arrow[r, "g"] & \mathcal{M}_{rat}
\end{tikzcd}$$

this commutative diag. is $\mathbb{G}_m$-equiv.

$Gr_{\mathcal{M}} := \mathcal{M} \times_{\mathcal{M}_{rat}} Y$ — the comma category is an affine grassmannian, let's denote it $Gr_{\mathcal{M}}$. $Gr_{\mathcal{M}}$ is equipped with a $\mathbb{G}_m$-action

Special case $\mathcal{M} = Coh^d(X)$: the structure map $Gr_{X_Y, D, \varepsilon} \longrightarrow Y$ is $\mathbb{G}_m$-equivariant

Recall: $\forall\, T \in Aff_S$, $Gr_{X,D,\varepsilon}(T) = \left\{ \begin{array}{l} (\mathcal{F}, \psi),\ \mathcal{F} \text{ is } T\text{-pure of dim } d. \\ \qquad \text{s.t. } D_T \text{ is } \mathcal{F}\text{-regular} \\ \psi : \mathcal{E}_T \to \mathcal{F} \text{ s.t. } \psi|_{U_T} \text{ is an iso} \end{array} \right\}$

$\forall\, T \in Aff_S$, $Gr^{\leq N}_{X,D,\varepsilon}(T) = \left\{ \begin{array}{l} (\mathcal{F}, \psi) \text{ in } Gr_{X,D,\varepsilon}(T) \text{ s.t.} \\ \mathcal{E}_T \subset \mathcal{F} \subset \mathcal{E}_T(ND_T) \end{array} \right\}$

For $P \in \mathbb{Q}[x]$, Define:

$$\forall T \in \mathrm{Aff}_S \,, \quad \mathrm{Gr}^{P}_{X,D,\mathcal{E}}(T) = \left\{ \begin{array}{l} (\mathcal{F}, \psi) \text{ in } \mathrm{Gr}_{X,D,\mathcal{E}}(T) \text{ s.t.} \\ P_{\mathcal{F}|_{X_t}} = P \,, \ \forall t \in T \end{array} \right\}$$

$$\forall T \in \mathrm{Aff}_S \,, \quad \mathrm{Gr}^{\leq N, P}_{X,D,\mathcal{E}}(T) = \mathrm{Gr}^{P}_{X,D,\mathcal{E}} \cap \mathrm{Gr}^{\leq N}_{X,D,\mathcal{E}}$$

$\rightsquigarrow$ each Y-projective strata $\mathrm{Gr}^{\leq N,P}_{\mathcal{M}}$ is $\mathbb{G}_m$-stable.

$Y \setminus (0,0)$ is q-compact, so τ factors trought one of the strata

$$\begin{tikzcd} & \mathrm{Gr}^{\leq N,P}_{\mathcal{M}} \arrow[r, "\mathrm{Forget}"] \arrow[d, "\tau'"] & \mathcal{M} \\ Y \setminus (0,0) \arrow[ur, "\tau"] \arrow[r, hook, "j"', "\circ"] & Y & \end{tikzcd}$$

$\Sigma := \overline{\tau(Y \setminus (0,0))}$ the scheme-closure of $Y \setminus (0,0)$ in $\mathrm{Gr}^{\leq N,P}_{\mathcal{M}}$.

$\Sigma \longrightarrow Y$ is projective (since $\mathrm{Gr}^{\leq N,P}_{\mathcal{M}}$ is projective over Y and Σ is closed subscheme of $\mathrm{Gr}^{\leq N,P}_{\mathcal{M}}$) so proper.

$\Sigma \xrightarrow{\tau'} Y$ is $\mathbb{G}_m$-equiv. by construction, and restricts to an iso

$\Sigma_{Y \setminus (0,0)} \xrightarrow{\cong} Y \setminus (0,0)$. $\rightsquigarrow (M_1)$ ✓

$\Sigma \longrightarrow \mathrm{Gr}^{\leq N,P}_{\mathcal{M}} \longrightarrow \mathcal{M}$ restricts to $f : Y \setminus (0,0) \longrightarrow \mathcal{M}$

$\rightsquigarrow \tilde{\varphi} : [\Sigma / \mathbb{G}_m] \longrightarrow \mathcal{M}$ since everything is $\mathbb{G}_m$-equivariant

$$
\begin{tikzcd}
\left[\left(\Sigma_{\mathcal{Y}_{\mathcal{X}}} \setminus (0,0)\right) / \mathbb{G}_m\right] \arrow[d, "f"'] \arrow[dr, "\tilde{\varphi}"] \arrow[dr, phantom, "\bigcirc", shift right=6] & \\
\mathcal{X} \setminus (0,0) \arrow[r, "\varphi"'] & \mathcal{M}
\end{tikzcd}
$$

$\rightsquigarrow (M_2)$ ✓

we use the <u>lemma</u>: $\exists\, m \gg 0$ s.t. $L_m^\vee\big|_{Gr_{\mathcal{M}}^{\leq N,\beta}}$ is $\mathcal{Y}$-ample $\forall\, n \geq m$.

Let $a \in \mathbb{Z}_{\geq 1}$. Consider
$$
\begin{aligned}
\mathbb{G}_m \times \mathbb{P}^1_k &\longrightarrow \mathbb{P}^1_k \\
(t, [x:y]) &\longmapsto [t^{-a}x : y]
\end{aligned}
$$

the assumption we need to work with in the axiom $(M3)$ is

SPS $\mathbb{P}^1_k \longrightarrow \Sigma_{(0,0)}$ finite $\mathbb{G}_m$-equivariant.

So $\exists\, m(N,\beta)$, $\forall\, n \geq m$, $L_n^\vee\big|_{\mathbb{P}^1_k}$ is ample.

So $L_m\big|_{\mathbb{P}^1_k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-N_m)$, for some $N_m > 0$.

And $\mathrm{wt}\, \mathcal{O}_{\mathbb{P}^1_k}(-N_m)\big|_{\infty} \geq \mathrm{wt}\, \mathcal{O}_{\mathbb{P}^1_k}(-N_m)\big|_{0}$

$$\nu\left(\tilde{\varphi}\big|[\infty/\mathbb{G}_m]\right) = \frac{\mathrm{wt}\,(L_m|_\infty)}{\sqrt{b(\tilde{\varphi}|_{[\infty/\mathbb{G}_m]})}}$$

by definition:

$$\nu\left(\tilde{\varphi}\big|[0/\mathbb{G}_m]\right) = \frac{\mathrm{wt}\,(L_m|_0)}{\sqrt{b(\tilde{\varphi}|_{[0/\mathbb{G}_m]})}}$$

$\mathrm{wt}\,(L_m|_\infty) \geq \mathrm{wt}\,(L_m|_0)$ for $m \gg 0$.

$$\begin{tikzcd}
{[\mathbb{P}^1_k/\mathbb{G}_m]} \arrow[r] \arrow[d] & \Sigma_{(0,0)} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \arrow[dl, Rightarrow] \\
(B\mathbb{G}_m)_k \arrow[r, "{[a]}"', "\text{induced by } a^{th} \text{ power on } \mathbb{G}_m."] & {[(0,0)/\mathbb{G}_m]} \arrow[r, "g_{(0,0)}"'] & \mathcal{M}_{rat}
\end{tikzcd}$$

So $\exists$ 2-morph between $[\infty/\mathbb{G}_m] \xrightarrow{\tilde{\varphi}} \mathcal{M} \longrightarrow \mathcal{M}_{rat}$

and $[0/\mathbb{G}_m] \xrightarrow{\tilde{\varphi}} \mathcal{M} \longrightarrow \mathcal{M}_{rat}$

So by <u>lemma</u>, $b(\tilde{\varphi}|_{[\infty/\mathbb{G}_m]}) = b(\tilde{\varphi}|_{[0/\mathbb{G}_m]})$

$\boxed{///}$

§2. HN-Boundedness

: for the purpose of checking eligibility of ν to define a Θ-filtration, we maximize $\nu(f)$ among all filtrations of points in a bounded family, it's enough to check only a filtration f s.t. the associated graded $f|_0$ lies in some other possibly larger bounded family., this idea is captured in the following HN-Boundedness:

Def: A polynomial numerical invariant ν satisfies the HN-boundedness condition if: $\forall T \in \mathrm{Aff}_S$ Noetherian, $\forall g: T \to \mathcal{M}$, $\exists \mathcal{U}_T \overset{\text{open}}{\underset{\text{qc}}{\subset}} \mathcal{M}$, $\forall t \in T$ closed with $k(t) = k$, $\forall f: \Theta_k \to \mathcal{M}$ non-deg filtration of $g(t)$ with $\nu(f) > 0$, $\exists f' \in \mathrm{Filt}(g(t))$ non-deg. s.t. $\nu(f') \geq \nu(f)$ and $f'|_0 \in \mathcal{U}_T \subset \mathcal{M}$.

Prop: ν is HN-bounded. for $\Lambda\,\mathrm{Coh}^d(X)$, where

$$\Lambda\,\mathrm{Coh}^d(X) : (\mathrm{Sch}/S)^{op} \longrightarrow \text{Groupoids}$$

$$T \mapsto \left\{ \begin{array}{l} \text{Groupoid of } T\text{-pure sheaves of dim } d \text{ on } X_T \\ \text{equipped with a } \Lambda|_{X_T}\text{-module structure} \end{array} \right\}$$

(Here Λ is the sheaf of finitely-presented ring of differential operators on X relative to S in the sense of Simpson plus a condition that says roughly that the stack $\Lambda\,\mathrm{Coh}^d(X)$ is finitely presented over the base S.

__Proof:__

$$F : \Lambda\mathrm{Coh}^d(X) \xrightarrow{\text{Forget}} \mathrm{Coh}^d(X) \text{ is of finite type.}$$

We find $\mathcal{W}_T \underset{qc}{\overset{\text{open}}{\subset}} \mathrm{Coh}^d(X)$ s.t. $\forall f \in \mathrm{Filt}(\Lambda\mathrm{Coh}^d(X))$, $\exists f' \in \mathrm{Filt}(\Lambda\mathrm{Coh}^d(X))$ s.t. $\nu(f') \geq \nu(f)$ and $F(f'|_0) \in \mathcal{W}_T$. Taking $\mathcal{U}_T = F^{-1}(\mathcal{W}_T) \subset \Lambda\mathrm{Coh}^d(X)$ we're done.

$\forall f \in \mathrm{Filt}(\Lambda\mathrm{Coh}^d(X))$, $\exists f' \in \mathrm{Filt}(\Lambda\mathrm{Coh}^d(X))$ s.t. f' convex and $\nu(f') \geq \nu(f)$.

(This is easily seen by induction on the length of the filtration by subsheaves induced by Rees construction). So we can actually restrict our search for f' to convex filtrations.

need to prove $\rightsquigarrow$ $\mathcal{W}_T := \left\{ \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_m/\mathcal{F}_{m+1} : \exists t \in T \text{ s.t. } (\mathcal{F}_m)_{m\in\mathbb{Z}} \in \mathrm{Filt}(\mathcal{F}_t) \text{ is convex} \right\}$

bounded? (in the sense of boundedness of a geom. point in a stack)

enough $\rightsquigarrow$ $\exists C$ uniform lower bound s.t.

$$\forall (\mathcal{F}_m)_{m\in\mathbb{Z}} \in \mathrm{Filt}(\mathcal{F}_t) : \hat{\mu}(\mathcal{F}_m/\mathcal{F}_{m+1}) \geq C \quad (\hat{\mu}\text{: Mumford slope})$$

The fact that this is a sufficient condition for boundedness can be proved again by induction.

Convexity $\rightsquigarrow$ $\bar{P}_{\mathcal{F}_{(q-1)}} \geq \bar{P}_{\mathcal{F}_{(q-2)}/\mathcal{F}_{(q-3)}} \geq \dots \geq \bar{P}_{\mathcal{F}(0)/\mathcal{F}(1)}$ for the associated graded pieces

$$\rightsquigarrow \hat{\mu}(\mathcal{F}_{(q-1)}) \geq \hat{\mu}\left(\mathcal{F}_{(q-2)}/\mathcal{F}_{(q-3)}\right) \geq \dots \geq \hat{\mu}(\mathcal{F}_{(0)}/\mathcal{F}_{(1)}).$$

But $\mathcal{F}_{(0)}/\mathcal{F}_{(1)}$ is a pure quotient of $\mathcal{F}_t$ so $\hat{\mu}(\mathcal{F}_{(0)}/\mathcal{F}_{(1)}) \geq \hat{\mu}_{min}(\mathcal{F}_t)$

$\geq C$

where $\hat{\mu}_{min}(\mathcal{F}_t)$ is the minimal slope among the graded pieces of the Gieseker HN-filtration.

Since $\mathcal{F}_t$ runs over a bounded family so

$\blacksquare$

Th: [Halper – Leistner]: "Main theorem"

Let ν be a polynomial numerical invariant on $\mathcal{M}$ defined by a sequence of rational line bdls and a norm on graded points. Then

(1) ν defines a weak Θ-stratif of $\mathcal{M}$ iff it is strictly Θ-monotone and HN-bounded

(2) sps conditions of (1) satisfied, assume ν strictly S-monotone and $\mathcal{M}^{\nu\text{-ss}} = \coprod_c B_c$, B_c open bounded substacks.

Then $\mathcal{M}^{\nu\text{-ss}}$ has a separated good mod. space.

As an application, we can see that the stack $\Lambda Coh^d(X)_P^{\nu\text{-ss}}$ admits a separated good moduli space.

Cor: ν defines a Θ-stratification of $\Lambda Coh^d(X)$.

Denote $\Lambda Coh^d(X)^{\nu\text{-ss}} \subset \Lambda Coh^d(X)$ the open substack of ν-semistable points.

We also note another important consequence of the "Main theorem" above is that $\Lambda Coh^d(X)^{\nu\text{-ss}}$ admits a good mod space.

Th: sps S is a scheme over $\mathbb{Q}$, let $P \in \mathbb{Q}[n]$.

Then $\Lambda Coh^d(X)_P^{\nu\text{-ss}}$ admits a good mod space.

Proof: Recall:

$$\Lambda\, Coh^d(X)_P : (Sch/S)^{op} \longrightarrow \text{Groupoids}$$

$$T \mapsto \left\{ \begin{array}{l} \text{Groupoid of } T\text{- pure sheaves of dim } d \text{ on } X_T \\ \text{equipped with a } \Lambda|_{X_T} \text{- module structure} \\ \text{and s.t. all the fibers have Hilbert polynomial } P \end{array} \right\}$$

we need to check conditions of (2) from "Main theorem"

we already proved that ν is strictly Θ- monotone and strictly S- monotone so (1) is verified.

for other conditions in (2) it's enough to prove that $\Lambda Coh^d(X)_P^{\nu\text{-}ss}$ is quasi-compact.

Proof sketch: $F : \Lambda Coh^d(X)_P^{\nu\text{-}ss} \xrightarrow{\text{Forget}} Coh^d(X)$ is q-compact

enough $\rightsquigarrow$ $F(\Lambda\, Coh^d(X)_P^{\nu\text{-}ss}) \underset{\text{bounded}}{\subset} Coh^d(X)$?

By a similar type of argument that we used before

$\exists\, C$ a uniform upper bound s.t. $\hat{\mu}_{max}(\mathcal{F}) \leq C, \forall \mathcal{F} \in \Lambda Coh^d(X)_P^{\nu\text{-}ss}$.

enough $\rightsquigarrow$ The set of sheaves $\mathcal{G} := \{ \mathcal{F} \in Coh^d(X) \mid \hat{\mu}_{max}(\mathcal{F}) \leq C \}$ bounded?

$X \longrightarrow S$ is of finite presentation and S is quasi-compact

So we can reduce to the case when S Noetherian.
The boundedness follows then from a Th. of Langer
[Langer, Semistable sheaves in positive characteristic Th. 4.4].

$\blacksquare$

Thank you!

Support Slide:

Claim: $X^*(\mathbb{G}_m) \cong \mathbb{Z}$

Proof: Let $\alpha : \mathbb{Z} \longrightarrow X^*(\mathbb{G}_m) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$

$$n \longrightarrow (t \to t^n)$$

$m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \rightsquigarrow m^* : \mathcal{O}_{\mathbb{G}_m} \longrightarrow \mathcal{O}_{\mathbb{G}_m} \otimes \mathcal{O}_{\mathbb{G}_m}$

$$t \mapsto (t \otimes t)$$

So $m^*(t^n) = t^n \otimes t^n = (t \otimes t)^n = (m^*(t))^n$

So $\alpha(n) : \mathbb{G}_m \to \mathbb{G}_m$ is a morph of alg groups, so α is well defined.

$$t \to t^n$$

And $\alpha(n+m)(t) = t^{n+m} = (t^n)^m = \alpha(m) \circ \alpha(n)(t)$

so α is a group morph. it's clearly injective.

We show α is surj.

Let $\phi \in X^*(\mathbb{G}_m)$.

$\phi^*(t) \in k[t, t^{-1}] \Rightarrow \phi^*(t) = \sum_{|i| < m} a_i t^i$

$$\Rightarrow m^*(\phi^*(t)) = \phi^*(t) \otimes \phi^*(t)$$

$$\Rightarrow \sum_i a_i t^i \otimes t^i = \sum_{i,j} a_i a_j t^i \otimes t^j$$

So at most one a_i is non 0, say a_m,

but $\phi(1) = 1$ so $a_m = 1$ hence $\phi = \alpha(m)$. $\boxed{/\!/}$

see rather Constr. P.14

Def: Let $\mathcal{M}$ be a stack, let $p \in \mathcal{M}$.

Let $\gamma : (\mathbb{G}_m^q)_k \to \mathrm{Aut}(p)$ be a homomorph of k-groups w/ finite kernel.

Then, a <u>polynomial numerical invariant</u> is a function

$\nu_\gamma : \mathbb{R}^q \setminus \{0\} \to \mathbb{R}[n]$ s.t. :

(1) ν_γ in unchanged under field extension

(2) ν is locally constant in algebraic families.

(3) Given $\phi : (\mathbb{G}_m^w)_k \to (\mathbb{G}_m^q)_k$ with finite kernel, then $\nu_{\gamma\circ\phi} = \nu_\gamma|_{\mathbb{R}^w}$ along $\mathbb{R}^w \hookrightarrow \mathbb{R}^q$ induced by ϕ.

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skip

$\forall m \in \mathbb{Z}$, let $M_m, L_m \in \mathrm{Pic}(\mathrm{Coh}^d(X))$

Fix $T \in \mathrm{Sch}_S$, $f : T \to \mathrm{Coh}^d(X)$, represented by $\mathcal{F} \in \mathrm{Coh}^d(X_T)$

$g : T \to \mathrm{Coh}^d(X)_P$

So $\mathcal{F}$ is T-pure of dimension d.

Then $f^* M_m := \det R\pi_{T*}(\mathcal{F}(m))$, $g^* L_m := g^* M_m \otimes (g^* b_d)^{-\otimes \overline{P}_{\mathcal{F}}(m)}$

where $b_d := \bigotimes_{j=0}^{d} M_j^{(-1)^j \binom{d}{j}}$ is a line bundle.

Def: a subset B of geom. pts in a stack $\mathcal{X}$ is called bounded if it's contained in the image of $|T| \to |\mathcal{X}|$ for some finite type q-compact S-scheme T and morph. $T \to \mathcal{X}$.