Forschungsseminar "Komplexe Analysis": Program

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Introduction

As the new semester starts, we organize a study seminar with the goal of understanding newly developped methods for studying the structure of the moduli stack $\mathrm{Coh}(X)$ of coherent sheaves on a projective scheme X, which has far-reaching results in the theory of moduli spaces, non-abelian Hodge correspondence, minimal model program, hyperkähler geometry and beyond. This new method is called "infinite-dimensional GIT".

The main objective is to make it all the way to proving the following result:

Theorem 0.1. *(cf. [HLHJ21, Theorems 5.11, 5.19, 6.10 and 6.15]) Let X be a projective scheme of finite presentation over a scheme S. To any relatively ample line bundle $\mathcal{O}_X(1)$ on X, one can associate a polynomial-valued numerical invariant ν that defines a Θ-stratification of the moduli stack $\Lambda\,\mathrm{Coh}^d(X)$ (resp. a θ-stratification of $\mathrm{Pair}^d_{\mathcal{A}}(X)$). If S is proper over $\mathbb{Q}$, then the μ-semistable locus admits a good moduli space that is separated (respectively proper) over S.*

△

The objects coming into play are:

- $\mathrm{Coh}^d(X)_P$ the stack of pure d-dimentional sheaves of fixed Hilbert polynomial,
- Θ-stratifications and numerical invariants
- The affine grassmanians $\mathrm{Gr}_{\mathrm{X,D},\mathcal{E}}$.
- The geometric template
- Rational filling conditions

The seminar is divided into the following nine talks:

1 Moduli Problems of interest

Aim: introduce moduli spaces of sheaves in order to define stacks of interest in infinite GIT.
References: see [Góm01] for a quick introduction to stalks and [Alp21] for more details.

- Motivation: a short discussion about moduli problems, (co)representability of the moduli functor, moduli stacks [Alp21, Introduction].

- Algebraic spaces and stacks: introduce the étale site, define algebraic spaces and Artin stacks [Alp21, Section 2.1], [Góm01, Section 3].
- Background on moduli schemes of vector bundles [HL10, Sections 4.1 and 4.2]
- Define good moduli spaces and discuss their properties.[Alp13, Section 3.1]. Give an example of a good moduli space [HL21, Definition 5.5.1].
- Introduce the moduli stack of pure coherent sheaves $\mathrm{Coh}(X)$ (potentially motivate its study by formulating the non-abelian Hodge correspondence using $\mathrm{Coh}(X)$).

2 Structure of $\mathrm{Coh}^d(X)$ and definition of related variants

We focus on the open substack $\mathrm{Coh}(X)_P^d$ of pure d-dimensional coherent sheaves of fixed Hilbert polynomial P.

- Boundedness in classical GIT. Definition of boundedness for families of vector bundles [Sch08, Section 2.2.3]. Examples: [Sch08, Remark 2.2.3.3] [Hos15, Example 2.22]
- Discuss GIT construction of the projective moduli space of the open substack of semistable sheaves $\mathrm{Coh}^d(X)_P^{ss}$ [HL10, Section 4].
- Discuss stratification of the unstable sheaves by Harder-Narasimhan filtrations indexed by discrete numerical invariants (Gieseker semistability). (e.g. the semistable vector bundles are the building blocks of all vector bundles) [Sch08, Section 2.2].
- Define two more variants of this moduli problem:
  - The moduli stack $\Lambda\,\mathrm{Coh}(X)^d$ of pure coherent coherent modules of dimension d over a sheaf Λ of rings of differential operators on X, which we denote $\Lambda\,\mathrm{Coh}(X)$.
  - The moduli stack of pairs $\mathrm{Pair}_{\mathcal{A}}(X)$ parameterizing a pure coherent sheaf $\mathcal{F}$ of dimension d on X along with a homomorphism $\mathcal{A} \to \mathcal{F}$ from a fixed coherent sheaf $\mathcal{A}$.

3 θ-stability theory

- Motivation [HL14, Page 3].
- The framework of θ-stability theory:
  - Define filtrations, and key maps from the stack of graded and filtered points of an arbitrary stack $\mathfrak{X}$ $(\mathrm{ev}_1, \mathrm{ev}_0, \mathrm{gr})$ [HL21, Section 1]. Recall the corresponding Rees construction [HL21, Proposition 1.0.1]
  - Define θ-stratifications [HL21, Definition 2.1.1] and HN filtrations and motivate the definitions by [HL21, Example 0.0.1] [HL21, Lemma 2.1.4]
  - Define a numerical invariant and stability function [HL21, 4.1.1]. Give an example in formula inspired from [HL21, Example 4.1.17]

4 Reformulate the main theorem of GIT in the theory of θ-stability

State and prove [HL21, Theorems 5.5.8 and 5.5.10]

- In what sense does θ-stability satisfies the Hilbert-Mumford criterion
  - Since the affine grassmanians where this condition holds are not projective, will we get a quasi projective quotient?
  - Polarizing line bundle, linearization of the action [**10.1093/qmath/45.4.515**], projective quotient.

5 ∞-dimensional Vs Classical GIT

- Philosophy of ∞-dimensional GIT, its advantages over classical GIT, its drawback [HLHJ21, Section 1.1].
- The framework of ∞-dimensional GIT:
  - Define the pseudofunctor $\mathrm{Coh}^d(X)_{rat}$.
  - Define the affine grassmanian $\mathrm{Gr}_{\mathrm{X,D},\mathcal{E}}$ [HLHJ21, Definition 3.12] and prove its representability by a strict-ind scheme that is ind-projective over the base [HLHJ21, Proposition 3.13] through proving [HLHJ21, Lemmas 3.15 and 3.16].

6 ∞-dimensional GIT: the geometric template

- Define rational filling conditions [HLHJ21, Definition 4.5].
- Reduce rational filling conditions of $\mathrm{Pair}_A^d(X)$ and $\Lambda\,\mathrm{Coh}^d(X)$ to those of the stack of pure sheaves $\mathrm{Coh}^d(X)$ [HLHJ21, Lemma 4.6].
- Prove rational filling for $\mathrm{Coh}^d(X)$ [HLHJ21, Lemma 4.7].

7 ∞-dimensional GIT: Strict monotonicity

- Define polynomial numerical invariants [HLHJ21, Definition 2.24], its strict Θ-monotonicity and its strict S-monotonicity [HLHJ21, Definition 2.27].
- Prove strict Θ-monotonicity and strict S-monotonicity of the polynomial numerical invariant in [Theorem 4.8][HLHJ21] through [Proposition 3.19 and Corollary 3.26][HLHJ21].
- State the boostrap of [Theorem 4.8][HLHJ21] to the more general case in [Theorem 4.10][HLHJ21].

8 Θ-stratification of $\Lambda\,\mathrm{Coh}^d(X)$

- Define HN-Boundedness of a polynomial numerical invariant [HLHJ21, Definition 2.28].
- State a necessary and sufficient condition of a (weak)-Θ-stratification to be a Θ-stratification [HLHJ21, Lemma 2.1.6].
- Prove that $\Lambda\,\mathrm{Coh}^d(X)$ has a Θ-stratification in [HLHJ21, Theorem 5.11] through [HLHJ21, Proposition 2.26, Proposition 5.5].
- Prove boundedness of the semistable locus of $\mathrm{Coh}^d(X)_P$ in [HLHJ21] through [Theorems 2.19 and 2.26][HLHJ21]

9 Θ-stratification of $\mathrm{Pair}^d_{\mathcal{A}}(X)$ and beyond

- Prove [HLHJ21, Theorem 6.10] through [HLHJ21, Theorems 6.10, 2.26].
- How to obtain stratifications for moduli spaces of decorated sheaves in [Sch05] and [Sch08].

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References

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