RESEARCH SEMINAR PROGRAM: HITCHIN MAP FOR HIGHER DIMENSIONAL VARIETIES

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Abstract. In 1987, Hitchin considered the reduction of the Yang-Mills self-duality equations from dimension four to dimension two. Although there are no finite energy solutions on $\mathbb{R}^2$, imposing conformal invariance of the reduced self-dual Yang-Mills equations under translation in two directions, and using connections and Higgs fields, he managed to study the solution over compact Riemann surfaces. The moduli space of solutions to Hitchin's self-duality equations with fixed determinant connection turns out to be isomorphic to the moduli space of rank-2 stable Higgs bundles with fixed determinant of odd degree and trace-free Higgs field. Hitchin observed that these moduli spaces enjoy a very rich geometry. In particular they yield the central object of our study this seminar: when X is a smooth projective variety, the Hitchin morphism sends a Higgs field to its "characteristic polynomial". The objective of our research seminar is to study the higher dimensional Hitchin morphism following [CN20]. When X has dimension at least two, this morphism is not surjective in general. Recently, Chen-Ngô introduced the space of spectral data $\mathcal{B}_X$, a closed subscheme of the Hitchin base through which the Hitchin morphism factors. Moreover, they conjectured that $\mathcal{B}_X$ is the image of the Hitchin morphism. As an application of their results, they prove that in dimension two, the Hitchin map of a ruled surface or a non-isotrivial elliptic surface with reduced fibres are very closely related, which is another reincarnation of the non-abelian Hodge theory and the fact that the fundamental group of a ruled surface or a non-isotrivial elliptic surface is isomorphic to the fundamental group of the base curve. Last but not least, the technical background of the paper is rich, as it exhibits connections between the Hitchin morphism for higher dimensional varieties, the invariant theory of the commuting schemes, and Weyl's polarization theorem.

1. Talk 1: Hitchin map in dimension one

In this introductory talk, we recall the point of view, adopted by N.J. Hitchin in his study of moduli spaces of stable vector bundles over a Riemann surface X, namely that of symplectic geometry of their cotangent bundles. We brush through the original construction of a Hitchin system and we see it as a bundle in "almost" two ways, one of which is through a generalization of a theorem of Narasimhan and Seshadri ([Hit87b], Theorem 4.3), and the other is through the Hitchin map. The latter sends a point in the cotangent bundle of the moduli space of all holomorphic bundles on X to its "characteristic polynomial", [Hit87a]. We show that it is proper when the rank and degree are coprime and that it is an algebraically completely integrable Hamiltonian system with respect to the holomorphic symplectic form, with generic fiber a Prym variety corresponding to the spectral cover of the Riemann surface at the image point. We follow [Hit87b] section 8 to see how Hitchin defines spectral curves such that their deformation yields a linear system on the cotangent bundle of a Riemann surface.

2. Talk 2: Cameral covers in dimension one

We follow the steps of Donagi in [Don95] who described an abelianization procedure,i.e., an equivalence between principal Higgs bundles and some spectral data. In defining a cameral cover of our Riemann surface for each point of the Hitchin base, we see that a Hitchin fibre is isomorphic to a union of abelian varieties when the cameral cover is a smooth curve.

3. Talk 3: Hitchin's morphism in higher dimensions

We see how the datum of a point in the moduli space of Hitchin pairs is equivalent to the datum of a morphism to a quotient stack involving a product of universal classifying bundles.

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We then recall Chevalley's restriction theorem and Kostant's section for the Lie algebra of a connected reductive group with a fixed "épinglage". Chevaley's morphism induces a morphism of quotient stacks that will be the main ingredient of the "stacky" construction of the Hitchin morphism for higher dimensional varieties. After, we seek a sufficient condition for the Hitchin fibre over a point of the Hitchin base to contain at least a k-point for a finite field k. Moreover, we contruct a canonical action of the Picard stack over the Hitchin base scheme on the Hitchin fibres, an action for which the orbits are open and dense. Finally, we define an open subspace (denoted $\mathcal{A}_X^{\heartsuit}$) of the Hitchin base over which fibres of the Hitchin map are much better understood. For this, we follow [Ngô06] section 2 and [CN20] sections 2 and 4.

4. Talk 4: The universal spectral data morphism

We explain how we obtain the higher dimensional Hitchin morphism in the language of stacks starting from a Higgs bundle on a proper smooth algebraic variety of dimension d over a field k (sections 2 and 5, [CN20]). Instead of studying directly the Hitchin morphism for a given variety X, we see how the authors study it indirectly through a certain universal morphism, called the universal spectral data morphism linked with invariant functions on the scheme of commuting elements in the Lie algebra of the structure group of a principal bundle. The existence of such a morphism is important for the study of the Hitchin morphism, so we formulate properties (conjecture 3.1, [CN20]) and existence of such a morphism (conjecture 3.2, [CN20]). We eventually prove that the latter is true for $G = \mathrm{GL}_n$ (Theorem 3.3, [CN20]). Finally, we prove that the Hitchin morphism factors through a thickening of a closed subscheme $\mathcal{B}_X$ of the Hitchin base, of much lower dimension (Proposition 5.1, [CN20]). We state the conjecture that the resulting morphism (called the spectral data morphism) is surjective and we prove it for $G = \mathrm{GL}_n$ in dimension 2.

5. Talk 5: Spectral covers

We review the construction of the universal spectral cover for dimension one (section 6, [CN20]). Then, using Weyl's polarisation construction, we build a universal spectral covering of the Chow scheme classifying zero dimensional cycles of length n of $\mathbb{A}^d$ (Proposition 6.1, [CN20]). We then prove that any fibre over an open locus $\mathcal{B}_X^{\heartsuit}(k)$ of $\mathcal{B}_X(k)$ is isomorphic to the stack of maximal Cohen-Macaulay sheaves of generic rank one on the spectral cover corresponding to the fibre (Proposition 6.3, [CN20]).

6. Talk 6: Cohen Maccaulay spectral surfaces

In order to have a useful decsription of a fibre of the Hitchin morphism, we need to clarify how to construct a finite Cohen-Macaulayfication of the spectral cover corresponding to our fibre. This is not always clear, but can be implemented in the case of surfaces. We recall Serre's theorem on extending locally free sheaves across a closed subscheme of codimension 2 (Theorem 7.1, [CN20], or Proposition 7, [Ser66]). We then recall some facts about the Hilbert-Chow morphism and prove that the spectral cover over any point in $\mathcal{B}_X^{\heartsuit}$ admits a finite Cohen-Macaulayfication (Proposition 7.2, [CN20]). We then refine Proposition 6.3, [CN20] and see that any fibre over $\mathcal{B}_X^{\heartsuit}(k)$ is isomorphic to the stack of maximal Cohen-Macaulay sheaves of generic rank one over the the Cohen-Macaulay spectral surface, namely, over the Cohen-Macaulayfication of the spectral cover corresponding to our fibre which moreover contains a Picard stack acting on the Hitchin fibre.As a byproduct, we obtain that the Hitchin fibre is not empty (Theorem 7.3, [CN20]). We finally give a sufficient condition for this action to be free (Lemma 7.5 and Proposition 7.6, [CN20]).

7. Talk 7: Applications to ruled and elliptic surfaces

We focus on surfaces fibred X over a curve C. In the case where the fibration has only reduced fibres, we prove (Lemma 8.1, [CN20]) the existence of a map over the spectral cover over $b \in \mathcal{B}_C^{\heartsuit}$ which is isomorphic to the finite Cohen-Macaulayfication of Proposition 7.2, [CN20]. We deduce a sufficient condition for the latter finite Cohen-Macaulay surface to be normal (Corollary 8.3,[CN20]). In the case $X := C \times \mathbb{P}_1$, we do computations of the spectral cover and its finite Cohen-Macaulayfication, and we describe the Hitchin fibre over $b \in \mathcal{B}_C^{\heartsuit}$ (Example 8.4 [CN20]). Finally, in the case of a ruled surface or a non-isotrivial elliptic surface with reduced fibres, we show that the Hitchin map of the surface and the Hitchin map for the base curve are closely interrelated.

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