BRIDGELAND STABILITY AND MODULI SPACES: PROGRAM

AG KOMPLEXE ANALYSIS

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Introduction.

In this new iteration of the research seminar, we set ourselves the goal of understanding a framework for the study of moduli spaces of complexes of sheaves and other related problems.

One such framework is provided by Bridgeland's Stability conditions on triangulated categories. They have been introduced in [Bri02], with inspiration from work in string theory [Dou02]

The main objective for the seminar is two fold, first to study the geometry of the space of stability conditions via results like:

Theorem 0.1. *(Theorem 5.15 [*[Bri02]*]) The map $\mathscr{Z} : Stab(X) \to Hom(\Lambda, \mathbb{C})$ given by $(\mathscr{A}, Z) \mapsto Z$ is a local homeomorphism. In particular, $Stab(X)$ is a complex manifold of dimension $rank(\Lambda)$, where Λ is the lattice obtained as the image of the map $\mathscr{Z}$ restricted to the Grothendieck group of the category $\mathscr{A}$.*

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Moreover to study the wall and chamber structure of this space. By this we are referring to the fact that the topology on $Stab(X)$ is defined in such as way that the variation of stable objects via variation of stability condition , happens in a controlled way. More specifically we would like to the following proposition:

Proposition 0.1. *(proposition 5.27* [Bri02]*) Let $v_0 \in \Lambda$ be a primitive class, and let $S \subset D^b(X)$ be an arbitrary set of objects of class v_0. Then there exists a collection of walls $W_w^S(v_0)$, $w \in \Lambda$, with the following properties:*

1. *Every wall $W_w^S(v_0)$ is a closed submanifold with boundary of real codimension one.*
2. *The collection $W_w^S(v_0)$ is locally finite (i.e., every compact subset $K \subset Stab(X)$ intersects only a finite number of walls).*
3. *For every stability condition (P, Z) on a wall in $W_w^S(v_0)$, there exists a phase $\phi \in \mathbb{R}$ and an inclusion $F_w \hookrightarrow E_{v_0}$ in $P(\phi)$ with $v(F_w) = w$ and $E_{v_0} \in S$.*
4. *If $C \subset Stab(X)$ is a connected component of the complement of $\bigcup_{w\in\Lambda} W_w^S(v_0)$ and $\sigma_1, \sigma_2 \in C$, then an object $E_{v_0} \in S$ is σ_1-stable if and only if it is σ_2-stable.*

The second objective of the seminar is the construction of a moduli space determined by fixing a stability condition and to study general properties that it might have such as properness or being of finite type over $\mathbb{C}$. Off course precise examples will be extremely important here if other properties, such as projectiveness of the moduli space, want to be achieved.

The structure of the seminar will be borrowed from the main reference [[MS16]] and supplemented by the secondary reference [[Dou02]]. That is to say six talks will be presented. Where the first one will be devoted to the motivating examples of stability for vector bundles on a curve and its immediate generalization to stability on torsion free sheaves on higher dimensional varieties. In the second talk the derived technology will be presented in order to define stability functions and the heart of a bounded t-structure on a triangulated category. In the third talk Bridgeland's Deformation Theorem (i.e 0.1), together with a sketch of its proof will be presented. For the fourth talk, since Bridgeland stability is not a priori associated to a GIT problem, then there is not a systematic approach to construct a moduli space for a given Bridgeland stability condition, thus some examples will be presented and discussed to see what kind of techniques can be used to build such moduli spaces in some cases. For the fifth talk the Wall and Chamber Structure on $Stab(X)$ will be constructed. And finally, for the sixth talk the birational geometry of moduli spaces of sheaves on surfaces will be tackled by using wall crossing techniques in Bridgeland's theory. Typical question are what their nef and effective cones are and what the stable base locus decomposition of the effective cone is, a concrete example of these techniques will be presented. Namely, Let X be a smooth complex projective surface of Picard rank one. We will deal with computing the largest wall for ideal sheaves of zero dimensional schemes $Z \subset X$. The moduli space of these ideal sheaves turns out to be the Hilbert scheme of points on X. The motivation for this problem lies in understanding its nef cone. More specifically we will have the following:

First Talk: Motivating examples. Let X be a smooth algebraic variety over the complex number field $\mathbb{C}$ with dimension n larger than one. For fixed c_1 in $Pic(X)$, c_2 in $A^2_{num}(X)$ which is the Chow group of codimension-two cycles on X modulo numerical equivalence and a polarization L on X, let $\mathscr{M}_L(c_1, c_2)$ be the moduli space of locally free rank-two sheaves stable with respect to L in the sense of Mumford-Takemoto such that their first and second Chern classes are c_1 and c_2 respectively. In this talk, we consider the problem: what is the difference between $\mathscr{M}_L(c_1, c_2)$ and $\mathscr{M}_{L'}(c_1, c_2)$ where L and L' are two different polarizations. The answer to this question can be treated by developing a theory about equivalence classes, walls and chambers of type (c_1, c_2) for polarizations on X. This is done in Chapter I of [Qin93]. The first two chapters of [MS16] can be used to supplement the basics on slope and Gieseker semistability.

A dictionary that translates which object corresponds to the space $Stab(X)$, which function corresponds to the stability condition for this example will be provided.

Talk 2: The derive category of coherent sheaves: Hearts and t-structures. Bridgeland stability is a direct generalization of slope stability. The main difference is that we will need to change the category we are working with. Coherent sheaves will never work in dimension ≥ 2. Instead, we will look for other abelian categories inside the bounded derived category of coherent sheaves on X.

To this end, we first have to treat the general notion of slope stability for abelian categories. Then we will introduce the notion of a bounded t-structure and speak about the heart of such a structure. Chapter 4 and section 5.1 of [MS16] treat this subjects, for the general theory of t-structures we refer to [BBD82].

Talk 3:Bridgeland's deformation theorem. We introduce the two equivalent definitions of Bridgeland's Deformation Theorem i.e.0.1. This is the content of section 5.2 and 5.3 of [MS16] for more details we will refer to the original reference [Bri02]

Talk 4: Moduli spaces. The main motivation for stability conditions is the study of moduli spaces of semistable objects. In this talk we recall the general theory of moduli spaces of complexes, and then define in general moduli spaces of Bridgeland semistable objects, this will be taken from section 5.4 of [MS16]. A variety of examples where the moduli spaces parametrizing $S-$equivalence classes of $\sigma-$semistable objects exists and is projective will be presented such as the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ found in [AM15] and the case of the Hilbert scheme of n points on $\mathbb{P}^2$ found in [ABCH12], it is worth noting that the construction of these moduli spaces passes through moduli spaces of quiver representations using G.I.T. But nevertheless there is a precise correspondence between wall-crossings in the Bridgeland stability manifold and wall-crossings between Mori cones. Which we hope will later give a concrete example of wall crossing phenomena.

Talk 5: Wall and Chamber Structure. For this talk we will explain how stable object change when the stability condition is varied within $Stab(X)$ itself. The entire talk is going to revolve around section 5.5 of the main reference [MS16]

Talk 6: In this talk we introduce the fundamental operation of tilting for hearts of bounded t-structures and use it to give examples of Bridgeland stability conditions on surfaces. Another key ingredient is the Bogomolov inequality for slope semistable sheaves, which we will recall as well. We then conclude with a few examples of stable objects, explicit description of walls, and the behavior at the large volume limit point. To do this the entire section 6 of [MS16] will guide the study, special attention will be given to section 6.4 and just the main ideas and theorems of section 6.3 will be presented.

References

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